TUTTE PATHS AND EVEN COVERS

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TUTTE PATHS AND EVEN COVERS

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To my parents

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TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

Tutte paths have been well studied in the literature due to their applications with the Hamiltonian cycle problem. We prove the existence of Tutte paths in circuit graphs in which the number of nontrivial bridges is bounded. As a consequence, we obtain sharp circumference bounds for essentially 4-connected planar graphs. The Traveling Salesperson Problem is a foundational problem in the optimization literature and generalizes the Hamiltonian cycle problem. Motivated by the Traveling Salesperson Problem, we investigate even covers of subcubic graphs, i.e., finding a small number of cycles that cover the majority of the vertices. As an application, we obtain a $5/4$-approximation algorithm for the Traveling Salesperson Problem on 2-connected cubic graphs.

CHAPTER 1
INTRODUCTION

1.1 Background and History

A *Hamiltonian cycle* in a graph is a cycle that passes through every vertex of a graph. The study of Hamiltonian cycles has its origin in the 19th Century mathematician, Sir William Rowan Hamilton. Hamilton studied a noncommutative algebra which characterized Hamiltonian cycles in the graph of the regular dodecahedron [16, 31, 32]. This algebra had a geometric interpretation of walking along the faces of the icosahedron, which Hamilton called the the icosian calculus. Hamilton branded this as a game, selling it to a game designer. This later proved to be a financial failure for the game maker [9]. While the name of such cycles traces back to Hamilton, it should be noted that Karper investigated the Hamiltonian cycle problem for more general polyhedra a few years prior to Hamilton [43].

Regardless of its origins, the study of Hamiltonian cycles has had profound impacts on mathematics, computer science, and optimization. The Four Color Theorem [4, 41] (also [52]) states that every planar graph can be face 4-colorable. For quite a long time, the Four Color Theorem was the Four Color Conjecture. The question was first asked as early as the 1850s [9, 30]. The conjecture, at first receiving little attention, eventually developed a notoriety of difficultly, accumulating a large list of ideas and techniques towards possible proof strategies [50, 53]. Many of these ideas, while failing their initial goal to prove the Four Color Theorem, matured into deep and rich combinatorial theories. It is well known now that if a plane graph is Hamiltonian, then it has a face 4-coloring. This connection between cycles and face-coloring can be traced as one of the early justifications for the serious academic study of Hamiltonian cycles, see for example [65]. Remarkably, even today, every proof of the Four Color Theorem has required computer assistance.

After giving a false proof the Four Color Theorem, Tait [58] conjectured that every 3-connected planar cubic graph contains a Hamiltonian cycle. If true, such a claim would also imply the Four Color Theorem. Tutte [64] disproved this conjecture by giving a counterexample. Tutte's construction relied on a gadget obtained from modifying the graph of a pentagonal prism, now called a Tutte fragment in the literature. There has been additional work on finding more counterexamples to Tait's conjecture, see [2] or [33].

Strengthening the connectivity assumption to Tait's conjecture is sufficient for the existence of Hamiltonian cycles. Whitney [65] proved that every 4-connected planar triangulations are Hamiltonian and Tutte [63] further generalized this by showing all 4-connected planar graphs are Hamiltonian. Thomassen [62] later showed that 4-connected planar graphs are Hamiltonian connected, i.e., there exists a Hamiltonian path between any two specified vertices. A small correction to Thomassen's proof was made by Chiba and Nishizeki [14] and a shorter alternative proof was given by Ozeki [51]. A slight improvement to Thomassen's result was given by both Sanders [54] and Thomas and Yu [59]. Algorithmic versions of these results have also been studied [8, 55].

We have seen that 4-connected planar graphs are Hamiltonian, and 3-connected planar graphs may not be. This leads to the natural question of understanding the relationship between connectivity and circumference of planar graphs. (The *circumference* of a graph is the length of a longest cycle in that graph.) For 2-connectivity, planar n-vertex graphs of size at least $n \geq 5$ may have a circumference as low as 4, with the infinite family of complete bipartite graphs $K_{2,n-2}$ exhibiting this bound. Chen and Yu [13] showed that the circumference of a 3-connected planar n-vertex graph is $\Theta(n^{\log_3 2})$. This bound too is best possible with Chen and Yu [13] giving a family of iterated planar triangulations that exhibit this bound.

For any positive integer k, a graph is *essentially k-connected* if it is $(k-1)$-connected and, for any $S \subseteq V(G)$ with $|S| < k$, $G - S$ is connected or has exactly two components one of which is trivial. A graph is *cubic* if all of its vertices have degree 3. Grünbaum and Malkevitch [29] observed that essentially 4-connected cubic planar n-vertex graphs have circumference at least $3n/4$, with Zhang [70] improving their bound by an additive constant of one. Jackson and Wormald [34] proved that the circumference of an essentially 4-connected n-vertex graph is at least $(2n+4)/5$. Using a set of discharging rule, Fabrici, Harant, Mohr, and Schmidt [21] improved this to $5(n+2)/8$. Fabrici, Harant, Mohr, and Schmidt [20] also showed that essentially 4-connected triangulations have circumference at least $2(n+4)/3$ and conjectured this bound could be extended to all essentially 4-connected planar graph. Wigal and Yu [68] and independently Kessler and Schmidt [42] (using completely different methods) proved the following.

Theorem 1.1.1. *Let $n \geq 6$ be an integer and let G be an essentially 4-connected planar n-vertex graph. Then the circumference of G is at least $\lceil (2n+6)/3 \rceil$.*

This bound is best possible. Take a 4-connected triangulation T on k vertices, and inside each face of T add a new vertex and three edges from the new vertex to the three vertices in the boundary of that face. The resulting graph, say G, has $n := 3k-4$ vertices. Now take an arbitrary cycle C in G. For each $x \in V(C)$ with degree three in G, deleting x from C and adding the edge between the two neighbors of x in C, we obtain a cycle in T, say D. Then $|D| \leq k$; which implies $|C| \leq 2k$. Hence, the circumference of G is at most $2k = 2(n+4)/3 = \lceil (2n+6)/3 \rceil$.

The proof of Theorem 1.1.1 builds on the ideas and techniques from the proofs of Tutte [63] and Thomassen [62], finding a cycle C in a 2-connected graph G such that every component of $G - C$ has at most three neighbors on C. A cycle or path with such property is denoted as *Tutte*. Note the assumption that G is 4-connected implies that C must be a Hamiltonian cycle. If we instead assume that G is essentially 4-connected, then each component of $G - C$ would be a single vertex. Thus if we bound the number of components of $G - C$, then we also obtain a lower bound for the length of C. This was the main idea employed in [68] to obtain the sharp lower bound for the circumference of essentially 4-connected graphs.

Another closely related problem to the Hamiltonian cycle problem is the famous Travelling Salesperson Problem. The *Travelling Salesperson Problem* (TSP) asks for a spanning cycle of minimum length in an edge-weighted complete graph. The problem has remained a cornerstone to the fields of combinatorics, computer science, and optimization. The TSP is NP-hard, as it generalizes the Hamiltonian cycle problem, one of Karp's original examples in his seminal paper on NP-completeness [36]. In fact, is not possible to approximate the TSP within any constant factor of the optimum unless $P = NP$ [69]. Thus it seems unlikely for there to exist an efficient algorithm to solve this problem. Regardless, the importance of the problem in practical matters cannot be understated, and has continued to receive significant attention.

An important special case of the TSP which admits a constant factor approximation is the *metric TSP* in which the edge weights form a metric, a natural assumption for many applications. A further specialization of the metric TSP is the *graphic TSP* in which the edge weights form the distance function in some underlying connected graph G on the same vertex set. This is equivalent to finding a spanning closed walk, a *TSP walk*, in G with the minimum number of edges. We denote this minimum length by tsp(G).

The graphic TSP still contains the Hamiltonian cycle problem, and is thus NP-hard to solve exactly. A classic result of Christofides [15] and independently Serdyukov [7, 57] gives a $\frac{3}{2}$-approximation for the metric TSP. For many years, this had remained the best approximation ratio for any nontrivial special case of the metric TSP. The first improvement was made in 2005 by Gamarnik, Lewenstein, and Sviridenko [22] who gave a $(\frac{3}{2} - \frac{5}{389})$-approximation algorithm for the special case of the graphic TSP on 3-connected cubic graphs. Following this result, Gharan, Saberi, and Singh [26] gave a $(\frac{3}{2} - \epsilon)$-approximation algorithm for the general graphic TSP. Then Mömke and Svensson [46] gave a novel approach for a 1.461-approximation algorithm for the graphic TSP, which was shown to be in fact a $\frac{13}{9}$-approximation by Mucha [47]. Later, Sebő and Vygen [56] presented a new algorithm for an improved $\frac{7}{5}$-approximation for the graphic TSP. For the metric TSP, the

$\frac{3}{2}$ ratio was only very recently improved by Karlin, Klein, and Gharan [35] to $(\frac{3}{2} - \varepsilon)$ for some constant $\varepsilon > 10^{-36}$. While both the metric and graphic TSP allow for constant factor approximation, they remain APX-hard, i.e., unless $P = NP$, there does not exist polynomial-time approximation schemes. In particular, it is known that the metric and graphic TSPs are NP-hard to approximate within a $\frac{123}{122}$ and $\frac{185}{184}$-factor of the optimum respectively [37, 45].

A special case of graphic TSP extensively studied is when the input graph is planar. Arora et al. [5] showed that planar graphs with arbitrary positive edge weights have a polynomial time approximation scheme for the TSP. Klein [44] later improved this, showing for any fixed $\varepsilon > 0$, there exists a linear-time algorithm that finds a TSP tour within $(1+\varepsilon)$ of the optimum. In particular, under the assumption of planarity, the TSP is no longer APX-hard. From the structural viewpoint, Kawarabayashi and Ozeki [40] showed that 3-connected planar graphs have TSP tours of length at most $4(n-1)/3$, with this bound being tight.

Another special case of the graphic TSP, namely on subcubic graphs, has received significant attention (a graph is *subcubic* if all of its vertices have degree at most 3). Subcubic and cubic graphs are simple classes of graphs which retain the inapproximability of the metric TSP. Even when restricted to subcubic and cubic graphs, it remains NP-hard to approximate within a $\frac{685}{684}$ and $\frac{1153}{1152}$-factor respectively [38]. Furthermore, subcubic graphs are known to exhibit the worst-case behavior in a well-known conjecture from the 80's (see [27]), which asserts that the subtour elimination linear program relaxation for the metric TSP has an integrality gap of $\frac{4}{3}$. This $\frac{4}{3}$-integrality gap can be asymptotically realized by a family of subcubic graphs, see for example [6].

Note that a polynomial-time constructive proof of the $\frac{4}{3}$-integrality gap would yield a $\frac{4}{3}$-approximation algorithm for the TSP. Motivated by this, Aggarwal, Garg, and Gupta [1] gave a $\frac{4}{3}$-approximation for 3-connected cubic graphs. This approximation ratio was extended to 2-connected cubic graphs by Boyd et al. [10], and to 2-connected subcubic graphs by Mömke and Svensson [46]. The $\frac{4}{3}$ ratio was then slightly improved for cubic graphs to $(\frac{4}{3} - \frac{1}{61326})$ by Correa, Larreé, and Soto [17] and independently to $(\frac{4}{3} - \frac{1}{8754})$ by Zuylen [71], which was further improved to 1.3 by Candráková and Lukot'ka [12], and later to $\frac{9}{7}$ by Dvořák, Král', and Mohar [19].

Let G be a simple 2-connected subcubic graph. We write $n(G)$ to denote the number of vertices in G, and $n_2(G)$ to denote the number of degree 2 vertices in G. Dvořák, Král', and Mohar [19] showed that G has a TSP walk of length at most $\frac{9n(G)+2n_2(G)}{7} - 1$. They also constructed infinitely many subcubic (respectively, cubic) graphs whose minimum TSP walks have lengths $\frac{5n(G)+n_2(G)}{4} - 1$ (respectively, $\frac{5n(G)}{4} - 2$), and conjectured that $\frac{5n(G)+n_2(G)}{4} - 1$ is the correct bound. Wigal, Yoo, and Yu proved the following.

Theorem 1.1.2. *[66] Let G be a 2-connected simple subcubic graph. Then* $\text{tsp}(G) \leq \frac{5n(G)+n_2(G)}{4} - 1$. *Moreover, a TSP walk of length at most $\frac{5n(G)+n_2(G)}{4} - 1$ can be found in $O(n(G)^2)$ time.*

Note this provides a $\frac{5}{4}$-approximation algorithm for the graphic TSP on simple cubic graphs. We remark that our algorithm is purely combinatorial and deterministic. We also characterize the extremal examples of Theorem 1.1.2; that is, the 2-connected simple subcubic graphs G such that $\text{tsp}(G) = \frac{5n(G)+n_2(G)}{4} - 1$ (see Theorem 3.4.5). As pointed out by Dvořák et al. [19], Theorem 1.1.2 is false for non-simple graphs. This can be seen from the graph obtained from three internally disjoint paths between two vertices, each of length $2k + 1$, by the addition of parallel edges so that it becomes cubic.

1.2 Preliminaries

In order to prove Theorem 1.1.1, the strategy employed is to find a cycle C in a 2-connected planar graph G such that every component of $G - C$ has at most three neighbors on C and the number of components of $G - C$ is as small as possible. We now introduce the related concepts and terminologies. For general references, we refer to [18] for graph theory and [69] for approximation algorithms.

Let G be a graph and $H \subseteq G$. An *H-bridge* of G is the subgraph of G induced by an edge in $E(G) \setminus E(H)$ with both incident vertices on H or induced by the edges of G that are incident with one or two vertices in a single component of $G - H$. We use $\beta_G(H)$ to denote the number of H-bridges in G with at least 3 vertices. For any H-bridge B of G, a vertex in $V(B \cap H)$ is called an *attachment* of B on H. We say that H is a *Tutte subgraph* of G if every H-bridge of G has at most three attachments on H. Moreover, for any subgraph $F \subseteq G$, H is said to be an *F-Tutte* subgraph of G if H is a Tutte subgraph of G and every H-bridge of G containing an edge of F has at most two attachments on H. A *Tutte cycle* (respectively, *Tutte path*) is a Tutte subgraph that is a cycle (respectively, path).

For any positive integer k and any graph G, a k-separation in G is a pair (G_1, G_2) of subgraphs of G such that $|V(G_1 \cap G_2)| = k$, $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \emptyset$, and $G_i \not\subseteq G_{3-i}$ for $i = 1, 2$. A k-cut in G is a set $S \subseteq V(G)$ with $|S| = k$ such that there exists a separation (G_1, G_2) in G with $|V(G_1 \cap G_2)| = k$ and $G_i - G_{3-i} \neq \emptyset$ for $i = 1, 2$.

To state our technical result, we need further notation. Given a plane graph G and a cycle C in G, we say that (G, C) is a *circuit graph* if G is 2-connected, C is the outer cycle of G (i.e. C bounds the infinite face of G), and, for any 2-cut T in G, each component of $G - T$ must contain a vertex of C. Note as G is embedded into the plane, C has a clockwise orientation and a counterclockwise orientation. For any distinct elements $x, y \in V(C) \cup E(C)$, we use xCy to denote the subpath of C in clockwise order from x to y such that $x, y \notin E(xCy)$, and we say that xCy is *good* if G has no 2-separation (G_1, G_2) with $V(G_1 \cap G_2) = \{s, t\}$ such that x, s, t, y occur on xCy in order, $sCt \subseteq G_2$, and $|G_2| \geq 3$.

Moreover, let

$$\tau_{Gxy} = \begin{cases} 1, & xCy \text{ is not good,} \\ 1, & |\{x, y\} \cap E(C)| = 1 \text{ and } x \text{ and } y \text{ are incident,} \\ 1/2, & |\{x, y\} \cap E(C)| = 1 \text{ and } |xCy| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

We remark now these τ parameters are a rescaling of the τ parameters present in Wigal and Yu [68], see Corollary 2.2.9. If there is no danger of confusion, we may drop the reference to G. In nonrigorous terms, these τ parameters for particular choices of x and y will function as a measurement from how far a circuit graph (G, C) is from being 3-connected. Our main technical theorem for Tutte paths is the following.

Theorem 1.2.1. *Let (G, C) be a circuit graph and let $u, v \in V(C)$ be distinct and $e \in E(C)$, such that u, e, v occur on C in clockwise order. Then G has a C-Tutte path P between u and v such that $e \in E(P)$ and*

$$\beta(P) \leq (|P| - 6)/2 + \tau_{vu} + \tau_{ue} + \tau_{ev} \tag{1.1}$$

Theorem 1.2.1 is a strengthening of the technical theorem presented in [68]. In particular, the number of nontrivial bridges is now bounded in terms of the vertices in the path P, rather than the number of vertices of the graph G. In Corollary 2.2.9, we recover the original technical result in terms of bounding the number of nontrivial bridges in terms of the vertex count. As an application, in Theorem 2.3.1 we obtain sharp circumference bounds for essentially 4-connected graphs.

In Section 2.1, we consider some special cases for Theorem 1.2.1. In Section 2.2, we prove Theorem 1.2.1. In Section 2.3, we apply Theorem 1.2.1 providing sharp circumference bounds for essentially 4-connected planar graphs.

To prove our result on TSP walks in subcubic graphs, i.e., Theorem 1.1.2, we follow [19] in considering spanning subgraphs F of G in which every vertex has even degree. We call such a spanning subgraph F an *even cover*, and note that when G is a subcubic, F consists of vertex-disjoint cycles and isolated vertices. We let $c(F)$ denote the number of cycles in F and $i(F)$ denote the number of isolated vertices in F. The *excess* of F is defined to be

$$\text{exc}(F) = 2c(F) + i(F).$$

For a graph G, let $\mathcal{E}(G)$ denote the set of even covers of G, and define the *excess* of G as

$$\operatorname{exc}(G) = \min_{F \in \mathcal{E}(G)} \operatorname{exc}(F).$$

As an example, consider the graph Θ which consists of three internally disjoint paths between two vertices, each path with k vertices of degree 2. It is easy to see that every even cover may contain at most one cycle. As a result, the even cover consisting of a cycle and k isolated vertices obtain the minimum excess. Thus for $k \geq 1$,

$$\operatorname{exc}(\Theta) = 2 + k \leq \frac{(3k+2)+3k}{4} + 1 = \frac{n(\Theta)+n_2(\Theta)}{4} + 1,$$

with equality when $k = 1$ (in which case $\Theta \cong K_{2,3}$).

It is observed in [19] that if G is a subcubic graph, then there is an exact relation between $\operatorname{tsp}(G)$ and $\operatorname{exc}(G)$:

Proposition 1.2.2 (Dvořák et al. [19]). *Let G be a subcubic graph. Then*

$$\operatorname{tsp}(G) = \operatorname{exc}(G) - 2 + n(G). \tag{1.2}$$

Moreover, an even cover $F \in \mathcal{E}(G)$ can be converted into a TSP walk in G of length $\operatorname{exc}(F) - 2 + n(G)$ in linear time.

Thus, to prove Theorem 1.1.2, it suffices to show that

$$\operatorname{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + 1. \tag{1.3}$$

and that an even cover F of G satisfying this bound can be found in quadratic time.

To prove Theorem 1.1.2, we attack the problem in the following way. A natural strategy is given an edge $e \in E(G)$, to find an even cover F such that $e \in E(F)$ such that $\operatorname{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$ as in (1.3). This is not always possible, but in this case, one can find an even cover F' such that $e \notin E(F')$ and $\text{exc}(F')$ satisfies (1.3). A similar situation holds if one asks for an even cover F such that $e \notin E(F)$ and $\text{exc}(F)$ satisfies (1.3). To prove this behavior indeed occurs for even covers, an understanding of the structural properties of the extremal graphs for when (1.3) fails is necessary.

In Section 3.1, we develop our key definitions and state our accompaning technical theorem (Theorem 3.1.4) from which Theorem 1.1.2 will follow. In Section 3.2, we provide some technical lemmas on the structure of the extremal graphs for Theorem 3.1.4, which we call *θ-chains*. We complete the proof of Theorem 3.1.4 in Section 3.3. In Section 3.4, we characterize extremal graphs for Theorem 3.1.4. In Section 3.5, we give a quadratic-time algorithm that finds an even cover F in simple 2-connected subcubic graphs G with $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$.

We end this section with notation. For a positive integer k, let $[k] = \{1, \ldots, k\}$. If G and H are graphs, we write $G \cup H$ (respectively, $G \cap H$) to denote the union (intersection) of G and H. Let G be a graph. If S is a set of vertices or a set of edges, we let $G - S$ denote the subgraph of G obtained by deleting elements of S as well as edges incident with a vertex in S. When $S = \{s\}$ is a singleton, we simply write $G - s$. If H is a subgraph of G, we let $G - H := G - V(G \cap H)$. For a collection of 2-element subsets of $V(G)$, we write $G + S$ for the graph with vertex set $V(G)$ and edge set $E(G) \cup S$. However, for $x, y \in V(G)$ we use $G + xy$ to denote the graph obtained from G by adding a (possibly parallel) edge between x and y. For a subgraph $H \subseteq G$ and a set $S \subseteq V(G)$, we let $H + S$ denote the subgraph of G such that $V(H + S) = V(H) \cup S$ and $E(H + S) = E(H)$. For $S \subseteq V(G)$, we use $N(S)$ to denote the neighborhood of S in G. If $S = \{s\}$ is a singleton, we simply write $N(s)$. When $|N(S)| \in \{1, 2\}$, *suppressing* S means deleting S and adding a (possibly loop or parallel) edge between the vertices of $N(S)$. When $S = \{s\}$ is a singleton, *suppressing* s means suppressing $\{s\}$.

CHAPTER 2

TUTTE PATHS

In this chapter, we provide results on bounding the number of nontrivial bridges of Tutte paths in circuit graphs.

2.1 Special Cases

In this section, we prove some technical lemmas and special cases to Theorem 1.2.1. The following lemma is concerned with the base cases of the induction.

Lemma 2.1.1. *Let (G, C) be a circuit graph and let $u, v \in V(C)$ be distinct and $e \in E(C)$, such that u, e, v occur on C in clockwise order. If $e = uv$ or $|G| = 3$ then Theorem 1.2.1 holds. In particular, G has a C-Tutte path P between u and v such that $e \in E(P)$, and $\beta(P) = 1$ if $e = uv$ and $\beta(P) = 0$ if $e \neq uv$ and $|G| = 3$.*

Proof. As G is 2-connected, we have $|G| \geq 3$. First suppose $e = uv$. Then vCu is not good because of the 2-separation $(uCv, G - uv)$; so $\tau_{vu} = 1$. Moreover, since u, v are both incident with e, $\tau_{ue} = \tau_{ev} = 1$. Hence, $P := uv$ gives the desired C-Tutte path as $\beta(P) = 1 = (|P| - 6)/2 + \tau_{vu} + \tau_{ue} + \tau_{ev}$.

Now assume $e \neq uv$ and $|G| = 3$. Further assume by symmetry that u is not incident with e. Then $\tau_{vu} = 0$, $\tau_{ue} = 1/2$, and $\tau_{ev} = 1$. Hence, $P := C - uv$ gives the desired C-Tutte path as $\beta(P) = 0 = (|P| - 6)/2 + \tau_{vu} + \tau_{ue} + \tau_{ev}$. □

The following two lemmas are concerned with the existence of particular 2-cuts.

Lemma 2.1.2. *Suppose $n \geq 4$ is an integer and Theorem 1.2.1 holds for graphs on at most $n - 1$ vertices. Let (G, C) be a circuit graph on n vertices, $u, v \in V(C)$ be distinct, and $e \in E(C)$ such that u, e, v occur on C in clockwise order. If G has a 2-separation (G_1, G_2)*

such that $\{u,v\} \subseteq V(G_1)$, $\{u,v\} \not\subseteq V(G_2)$, $e \in E(G_2)$, *and* $|G_2| \geq 3$, *then* G *has a* C*-Tutte Path* P *between* u *and* v *such that* $e \in E(P)$ *and* $\beta(P) \leq (|P|-6)/2+\tau_{vu}+\tau_{ue}+\tau_{ev}$.

Proof. Let $V(G_1 \cap G_2) = \{x,y\}$ with $x \in V(eCv)$ and $y \in V(uCe)$. See Figure 2.1. Let $G_i' := G_i + xy$ for $i \in \{1,2\}$ such that G_1' is a plane graph with outer cycle $C_1 := xCy+yx$ and G_2' is a plane graph with outer cycle $C_2 := yCx+xy$, i.e., both (G_1', C_1) and (G_2', C_2) are circuit graphs. Let $e_1 := xy$. Since $\{u,v\} \not\subseteq V(G_2)$, we may assume by symmetry that $u \neq y$.

By assumption, G_1' has a C_1-Tutte path between u and v such that $e_1 \in E(P_1)$ and

$$\beta_{G_1'}(P_1) \leq (|P_1|-6)/2 + \tau_{G_1'vu} + \tau_{G_1'ue_1} + \tau_{G_1'e_1v},$$

and G_2' has a C_2-Tutte path P_2 between x and y such that $e \in E(P_2)$ and

$$\beta_{G_2'}(P_2) \leq (|P_2|-6)/2 + \tau_{G_2'xy} + \tau_{G_2'ye} + \tau_{G_2'ex}.$$

Let $P := (P_1 - e_1) \cup P_2$, and note P is a C-Tutte path in G between u and v such that $e \in E(P)$. Moreover, as $|P_1| + |P_2| = |P| + 2$, $\tau_{G_1'vu} = \tau_{Gvu}$, and $\tau_{G_2'xy} = 0$, we have,

$$\begin{aligned}\beta_G(P) &= \beta_{G_1'}(P_1) + \beta_{G_2'}(P_2)\\ &\leq (|P|-6)/2 - 2 + \tau_{Gvu} + \tau_{G_1'ue_1} + \tau_{G_1'e_1v} + \tau_{G_2'ye} + \tau_{G_2'ex}.\end{aligned}$$

We claim that $\tau_{G_1'e_1v} + \tau_{G_2'ex} \leq \tau_{Gev} + 1$. This is clear if $\tau_{Gev} = 1$. If $\tau_{Gev} = 1/2$ then $|eCv| = 2$, and, hence, $|e_1C_1v| = |xCv| = 2$ or $|eC_2x| = |eCx| = 2$; so $\tau_{G_1'e_1v} = 1/2$ or $\tau_{G_2'ex} = 1/2$, and the claim holds as well. Now suppose $\tau_{Gev} = 0$. Then $|eCv| \geq 3$. and eCv is good. So $|e_1C_1v| \geq 3$ and e_1C_1v is good in G_1', or $|eC_2x| \geq 3$ and $|eC_2x|$ is good in G_2', or $|e_1C_1v| = |eC_2x| = 2$. Hence, $\tau_{G_1'e_1v} = 0$, or $\tau_{G_2'ex} = 0$, or $\tau_{G_1'e_1v} = \tau_{G_2'ex} = 1/2$. In either case, the claim holds.

By similar argument, $\tau_{G_1'ue_1} + \tau_{G_2'ye} \leq \tau_{Gue} + 1$. So $\beta_G(P) \leq (|P|-6)/2 + \tau_{Gvu} +$

$\tau_{Gue} + \tau_{Gev}$. $\square$

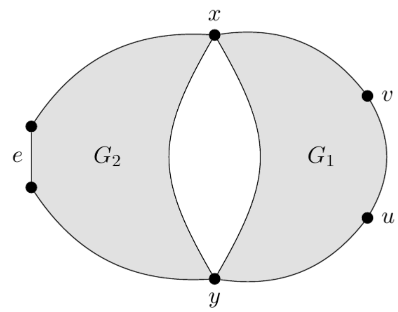

Figure 2.1: The separation (G_1, G_2) of G.

Lemma 2.1.3. *Suppose $n \geq 4$ is an integer and Theorem 1.2.1 holds for graphs on at most $n-1$ vertices. Let (G, C) be a circuit graph on n vertices, $u, v \in V(C)$ be distinct, and $e = xy \in E(C)$, such that u, x, y, v occur on C in clockwise order.*

If $\{u, x\}$ or $\{v, y\}$ is a 2-cut in G then G has a C-Tutte path P between u and v such that $e \in E(P)$ and $\beta_G(P) \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$.

Proof. Suppose $\{u, x\}$ or $\{v, y\}$ is a 2-cut in G, say $\{u, x\}$ by symmetry. Then G has a 2-separation (G_1, G_2) such that $xCu \subseteq G_1, uCx \subseteq G_2$, and $|G_2| \geq 3$. We choose (G_1, G_2) so that G_2 is maximal. See Figure 2.2. Note then $ux \notin E(G_1)$ and $\tau_{Gue} = 1$.

Case 1. G_1 is 2-connected.

Let C_1 denote the outer cycle of G_1. As (G, C) is a circuit graph, (G_1, C_1) is a circuit graph as well. By assumption, G_1 has a C_1-Tutte path P between u and v such that $e \in E(P)$ and

$$\beta_{G_1}(P) \leq (|P| - 6)/2 + \tau_{G_1vu} + \tau_{G_1ue} + \tau_{G_1ev}.$$

Note that $\tau_{G_1vu} = \tau_{Gvu}, \tau_{G_1ue} = 0$ (as $ux \notin E(G_1)$), and $\tau_{G_1ev} = \tau_{Gev}$. So

$$\beta_G(P) = \beta_{G_1}(P) + 1 \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$$

and thus P is the desired path.

Case 2. G_1 is not 2-connected.

Let $G_1' := G_1 + ux$ be the plane graph with outer cycle $C_1 := xCu + ux$, and let $G_2' := G_2 + xu$ be the plane graph with outer cycle $C_2 := uCx + xu$. As (G, C) is a circuit graph, we see that both (G_1', C_1) and (G_2', C_2) are circuit graphs. Note $\tau_{G_1'vu} = \tau_{Gvu}$, $\tau_{G_1'ue} = 1/2$, and $\tau_{G_1'ev} = \tau_{Gev}$. By assumption, G_1' has a C_1-Tutte path P_1 between u and v such that $e \in E(P_1)$ and

$$\begin{aligned}\beta_{G_1'}(P_1) &\leq (|P_1| - 6)/2 + \tau_{G_1'vu} + \tau_{G_1'ue} + \tau_{G_1'ev} \\ &= (|P| - 6)/2 + \tau_{Gvu} + (\tau_{Gue} - 1/2) + \tau_{Gev}.\end{aligned}$$

As G_1 is not 2-connected, $ux \in E(P_1)$.

Choose $e' \in E(uC_2x)$ such that $\tau_{G_2'e'x} = 1/2$ and $\tau_{G_2'ue'} \leq 1$. Note $\tau_{G_2'xu} = 0$. By assumption, G_2' has a C_2-Tutte path P_2 between x and u such that $e' \in E(P_2)$ and

$$\beta_{G_2}(P_2) \leq (|P_2| - 6)/2 + \tau_{G_2'xu} + \tau_{G_2'ue'} + \tau_{G_2'e'x} \leq (|P_2| - 6)/2 + 3/2$$

Now $P := (P_1 - ux) \cup P_2$ is a C-Tutte path in G between u and v such that $e \in E(P)$. Moreover,

$$\begin{aligned}\beta_G(P) &= \beta_{G_1'}(P_1) + \beta_{G_2'}(P_2) \\ &\leq (|P_1| - 6)/2 + \tau_{Gvu} + (\tau_{Gue} - 1/2) + \tau_{Gev} + (|P_2| - 6)/2 + 3/2 \\ &< (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.\end{aligned}$$

So P is the desired path. $\square$

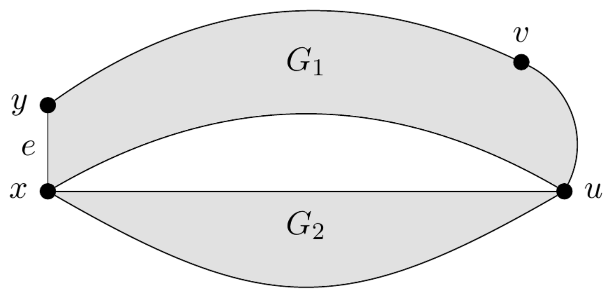

Figure 2.2: The other separation (G_1, G_2) of G.

The following lemma is included for later convenience where we want to find a Tutte path that includes three designated vertices. A vertex count analog to this lemma first appeared in [67].

Lemma 2.1.4. *Suppose $n \geq 3$ is an integer and Theorem 1.2.1 holds for graphs on at most $n-1$ vertices. Let (G, C) be a circuit graph on n vertices and $u, v, z \in V(C)$ be distinct such that u, z, v occur on C in clockwise order. Then G has a C-Tutte path P between u and v such that $z \in V(P)$ and $\beta(P) \leq (|P|-3)/2 + \tau_{vu}$.*

Proof. First suppose $n = 3$ and choose $e \in E(G)$ such that $e \neq uv$. Then by Lemma 2.1.1, then G has a C-Tutte path P between u and v such that $\beta(P) = 0$. In particular, P is Hamiltonian and thus $z \in V(P)$. We proceed with induction on n.

Now suppose $n > 3$ and there is no 2-cut separating z from $\{u, v\}$. Choose an edge e such that $\tau_{Gue} \leq 1/2$. By induction there exists a C-Tutte path P between u and v such that $e \in V(P)$ and

$$\begin{aligned} \beta_G(P) &\leq (|P|-6)/2 + \tau_{Gue} + \tau_{Gev} + \tau_{Gvu} \\ &\leq (|P|-3)/2 + \tau_{Gvu}. \end{aligned}$$

As there is no 2-cut separating z from $\{u, v\}$, we have $z \in V(P)$.

Now let (G_1, G_2) be a 2-separation in G with $\{u, v\} \subseteq V(G_1)$ and $z \in V(G_2)$. Let $V(G_1 \cap G_2) = \{x, y\}$ such that u, x, z, y, v appear on C in clockwise order. Let $G_1' :=$

$G_1 + xy$ and $G_2' := G_2 + xy$ be plane graphs with respective outer cycles $C_1 := yCx + xy$ and $C_2 := xCy + xy$. Then (G_i', C_i) are circuit graphs for $i \in [2]$.

As $|G_1'| < n$, Theorem 1.2.1 holds for G_1' (by assumption), thus G_1' has a C_1-Tutte path P_1 between u and v such that $e := xy \in E(P_1)$ and

$$\begin{aligned}\beta_{G_1'}(P_1) &\leq (|P_1| - 6)/2 + +\tau_{G_1'ue} + \tau_{G_1'ev} + \tau_{G_1'vu} \\ &\leq (|P_1| - 2)/2 + \tau_{G_1'vu}.\end{aligned}$$

Note that $\tau_{G_2'yx} = 0$. Thus, by induction, G_2' has a C_2-Tutte path P_2 between x and y such that $z \in V(P_2)$ and

$$\beta_{G_2'}(P_2) \leq (|P_2| - 3)/2.$$

Let $P := (P_1 - xy) \cup P_2$. As $|P_1| + |P_2| = |P| + 2$, we have

$$\begin{aligned}\beta_G(P) &= \beta_{G_1'}(P_1) + \beta_{G_2'}(P_2) \\ &\leq (|P_1| - 2)/2 + \tau_{G_1'vu} + (|P_2| - 3)/2 \\ &= (|P| - 3)/3 + \tau_{Gvu}.\end{aligned}$$

As $z \in V(P)$, we have constructed the desired Tutte path. $\square$

2.2 Proof of Theorem 1.2.1

We apply induction on $n = |G|$. By Lemma 2.1.1 and by symmetry, we may assume that u is not incident with e and $|G| = n \geq 4$, and the assertion holds for graphs on at most $n - 1$ vertices. Let $e = v'v''$ such that u, v', v'', v occur on C in clockwise order. We may conclude the following by Lemma 2.1.3.

Claim 2.2.1. *Neither $\{u, v'\}$ nor $\{v', v''\}$ is a cut in G.*

Moreover, by Lemma 2.1.2, we may assume that G has no 2-cut $T \neq \{u, v\}$ separating e from $\{u, v\}$. Thus, by planarity, uCe is contained in a block of $G - eCv$, which is denoted by H. (See Figure 2.3) Note that $H \cong K_2$ or H is 2-connected.

Claim 2.2.2. *H is 2-connected.*

Proof. For, suppose that $H \cong K_2$. Note that v' must have degree 2 in G and $G - v'$ is 2-connected; for otherwise, there exists a vertex $z \in V(v''Cv)$ such that $\{v, z\}$ is a 2-cut in G separating e from $\{u, v\}$, contradicting Lemma 2.1.2. Let $C' := v''Cu + uv''$ be the outer cycle of $G' := (G - v') + uv''$, and let $e' := uv''$. Note that (G', C') is a circuit graph, $\tau_{G'ue'} = 1 = \tau_{Gue} + 1/2$, $\tau_{G'e'v} = \tau_{Gev}$, and $\tau_{G'vu} = \tau_{Gvu}$. Hence, by the induction hypothesis, G' has a C'-Tutte path P' between u and v such that $e' \in E(P')$ and

$$\beta_{G'}(P') \leq (|P'| - 6)/2 + \tau_{G'vu} + \tau_{G'ue'} + \tau_{G'e'v}$$

Now $P := (P' - e') \cup uv'v''$ is a C-Tutte path in G between u and v such that $e \in E(P)$ and $\beta_G(P) = \beta_{G'}(P') \leq (|P'| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev} = (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}$. ∎

By Claim 2.2.2, let D denote the outer cycle of H. Our strategy is to use the induction hypothesis to find a path in H and extend it to the desired path in G along eCv. Let $w \in V(vCu)$ such that $wDv' = wCv'$ and, subject to this, wCv' is maximal. By Lemma 2.1.4, we have the following claim.

Claim 2.2.3. *H contains a D-Tutte path P_H between u and v' such that $w \in V(P_H)$ and*

$$\beta_H(P_H) = (|P_H| - 6)/2 + \tau_{Gue} + 3/2.$$

We wish to extend P_H along eCv to the desired path P in G. Thus we need a useful description of the structure of the part of G that lies between H and eCv. See Figure 2.3.

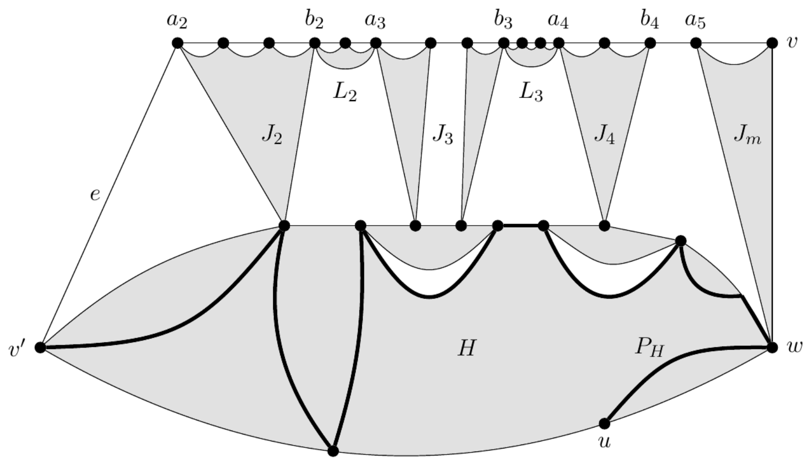

Figure 2.3: The subgraph H of G and the bridges between eCv and H

Let $\mathcal{B}$ be the the set of $(H \cup eCv)$-bridges of G. Then $G = H \cup eCv \cup (\bigcup_{B\in\mathcal{B}} B)$. As H is a block of $G - eCv$, $|B \cap H| \leq 1$ for all $B \in \mathcal{B}$.

For $B_1, B_2 \in \mathcal{B}$ with $|B_1 \cap H| = 1 = |B_2 \cap H|$, we denote by $B_1 \sim B_2$ if $V(B_1 \cap H) = V(B_2 \cap H) \subseteq V(P_H)$, or if there exists a P_H-bridge B of H such that $V(B_1 \cap H) \cup V(B_2 \cap H) \subseteq V(B - P_H)$. Clearly, $\sim$ is an equivalence relation on $\mathcal{B}$. Let $\mathcal{B}_i$, $i \in [m]$, be the equivalence classes of $\mathcal{B}$ with respect to $\sim$, such that $H \cap \big(\bigcup_{B\in\mathcal{B}_i} B\big)$, $i \in [m]$, occur on D in order from v' to w, with $v' \in V(B)$ for all $B \in \mathcal{B}_1$ and $w \in V(B')$ for all $B' \in \mathcal{B}_m$. Let $a_i, b_i \in V(eCv)$ such that

(a) $a_i \in V(B)$ for some $B \in \mathcal{B}_i$ and $b_i \in V(B')$ for some $B' \in \mathcal{B}_i$ (possibly $B = B'$),

(b) v'', a_i, b_i, v occur on eCv in order, and

(c) subject to (a) and (b), a_iCb_i is maximal.

Note that $v'' = a_1$ and $b_m = v$. Let J_i denote the union of a_iCb_i, all members of $\mathcal{B}_i$, those $(H \cup eCv)$-bridges of G whose attachments are all contained in a_iCb_i, and the P_H-bridge of H containing $B \cap H$ for all $B \in \mathcal{B}_i$.

For $1 < i < m$, let L_i denote the union of b_iCa_{i+1} and those $(eCv \cup H)$-bridges of G whose attachments are all contained in b_iCa_{i+1}. Note that $|J_i \cap P_H| \in \{1, 2\}$ for all $1 \leq i \leq m$. Let

- $\mathcal{J}_1$ be the set of all J_i, $1 < i < m$, such that $|J_i \cap P_H| = 1$ and $a_i \neq b_i$,

- $\mathcal{J}_2$ be the set of all J_i, $1 < i < m$, such that $|J_i \cap P_H| = 2$, and

- $\mathcal{L}$ be the set of all L_i, $1 < i < m$.

By Lemma 2.1.2, we have $|J_1| = 2$. Thus we have the following claim.

Claim 2.2.4. *J_1 has a path $P_1 = J_1$ such that $\beta_{J_1}(P_1) = 0 = (|J_1| - 1)/2 - 1/2$.*

Claim 2.2.5. *For $J_i \in \mathcal{J}_1$, J_i has an a_i-b_i path P_i such that $P_i \cup (J_i \cap P_H)$ is an a_iCb_i-Tutte subgraph of J_i and*

$$\beta_{J_i}(P_i \cup (J_i \cap P_H)) \leq \begin{cases} (|P_i| - 1)/2 - 1, & \text{if } eCv \text{ is good and } |a_iCb_i| \geq 3, \\ (|P_i| - 1)/2 - 1/2, & \text{if } eCv \text{ is good and } |a_iCb_i| = 2, \\ (|P_i| - 1)/2, & \text{otherwise.} \end{cases}$$

Proof. Let $V(J_i \cap P_H) = \{x\}$. Consider the plane graph $J_i' := J_i + a_ix$ with outer cycle C_i consisting of a_iCb_i, the edge $e_i := xa_i$, and the path in the outer walk of J_i between b_i and x not containing a_i. Then (J_i', C_i) is a circuit graph. Note that $\tau_{J_i'xe_i} = 1$ and $\tau_{J_i'b_ix} = 0$.

Hence, by the induction hypothesis, J_i' has a C_i-Tutte path P_i' between x and b_i such that $e_i \in E(P_i')$ and $\beta_{J_i'}(P_i') \leq (|P_i'| - 6)/2 + \tau_{J_i'e_ib_i} + 1$. Note that $\tau_{J_i'e_ib_i} \leq 1$. If eCv is good, then $\tau_{J_i'e_ib_i} \leq 1/2$ (as $a_i \neq b_i$), and $\tau_{J_i'e_ib_i} = 0$ if $|a_iCb_i| \geq 3$. Let $P_i := P_i' - x$. As $|P_i'| = |P_i| + 1$, P_i gives the desired path. ∎

Claim 2.2.6. *For $J_i \in \mathcal{J}_2$, J_i has an a_i-b_i path P_i such that $P_i \cup (J_i \cap P_H)$ is an a_iCb_i-Tutte subgraph of J_i' and*

$$\beta_{J_i}(P_i \cup (J_i \cap P_H)) \leq \begin{cases} (|P_i| - 1)/2, & \text{if } eCv \text{ is good and } a_i \neq b_i, \\ (|P_i| - 1)/2 + 1, & \text{otherwise.} \end{cases}$$

Proof. If $a_i = b_i$, we let $P_i = a_i$ which gives the desired path. Now suppose $a_i \neq b_i$. Let $V(J_i \cap P_H) = \{x, y\}$, let J'_i be the block of $J_i - \{x, y\}$ containing a_iCb_i, and let C_i be the outer cycle of J'_i with $a_iC_ib_i = a_iCb_i$. We may further assume that v', y, x, w occur on D in clockwise order.

By planarity there exists a vertex $z \in V(b_iC_ia_i) \setminus \{a_i, b_i\}$ such that $b_iC_iz - z$ contains no neighbor of y and $zC_ia_i - z$ contains no neighbor of x. By Lemma 2.1.4, J'_i contains a Tutte path P_i between a_i and b_i such that $z \in V(P_i)$ and

$$\beta_{J'_i}(P_i) \leq (|P_i| - 6)/2 + \tau_{J'_i a_i b_i} + 3/2.$$

As $\beta_{J_i}(P_i) \leq \beta_{J'_i}(P_i) + 1$, and $\tau_{J'_i a_i b_i} = 0$ if eCv is good, P_i is the desired Tutte path. ∎

Claim 2.2.7. *J_m has a path P_m between a_m and $b_m = v$ such that P_m+w is an a_mCw-Tutte subgraph of J_m and*

$$\beta_{J_m}(P_m + w) \leq \begin{cases} (|P_m| - 1)/2 - 1 + \tau_{Gvu}, & \textit{if } eCv \textit{ is good and } |a_mCb_m| \geq 3, \\ (|P_m| - 1)/2 - 1/2 + \tau_{Gvu}, & \textit{if } eCv \textit{ is good and } |a_mCb_m| = 2, \\ (|P_m| - 1)/2 + \tau_{Gvu}, & \textit{otherwise.} \end{cases}$$

Proof. First, suppose $a_m = b_m$. Then $P_m = a_m$ gives the desired path.

Now assume $a_m \neq b_m$ and consider the plane graph $J'_m := J_m + a_mw$ with outer cycle $C_m := a_mCw + a_mw$. Then (J'_m, C_m) is a circuit graph. Let $e_m := a_mw$. Note that $\tau_{J'_m b_m w} \leq \tau_{Gvu}$ and $\tau_{J'_m w e_m} = 1$.

Hence, by induction hypothesis, J'_m has a C_m-Tutte path between b_m and w such that $e_m \in E(P'_m)$ and

$$\begin{aligned} \beta_{J'_m}(P'_m) &\leq (|P'_m| - 6)/2 + \tau_{Gvu} + 1 + \tau_{J'_m e_m b_m} \\ &= (|P'_m| - 2)/2 - 1 + \tau_{Gvu} + \tau_{J'_m e_m b_m}. \end{aligned}$$

Note that $\tau_{J'_m e_m b_m} \leq 1/2$ (when eCv is good) and $\tau_{J'_m e_m b_m} \leq 1$ (when eCv is not good). Furthermore if eCv if good and $|a_m C b_m| \geq 3$, then $\tau_{J'_m e_m b_m} = 0$. Hence, $P_m = P'_m - w$ gives the desired path. ∎

Next, we consider those $(eCv \cup H)$-bridges of G with all attachments contained in eCv.

Claim 2.2.8. *L_i contains a $b_i C a_{i+1}$-Tutte path Q_i from b_i to a_{i+1} such that*

$$\beta_{L_i}(Q_i) \leq \begin{cases} (|Q_i|-1)/2 - 1, & \textit{if } |b_i C a_{i+1}| \geq 3, \\ (|Q_i|-1)/2 - 1/2, & \textit{if } |b_i C a_{i+1}| = 2, \\ (|Q_i|-1)/2, & \textit{otherwise.} \end{cases}$$

In particular, $\beta_{L_i}(Q_i) \leq \max\{0, (|Q_i|-1)/2 - 1\}$.

Proof. If $|b_i C a_{i+1}| \leq 2$ then let $Q_i := b_i C a_{i+1}$; we see that $\beta_{L_i}(Q_i) = 0$. So suppose $|b_i C a_{i+1}| \geq 3$. Then consider the plane graph $L'_i := L_i + b_i a_{i+1}$ with outer cycle $D_i := b_i C a_{i+1} + a_{i+1} b_i$. Note that (L'_i, D_i) is a circuit graph. Choose an edge $e'_i \in E(b_i C a_{i+1})$ so that $\tau_{L'_i b_i e'_i} = 1/2$. Note that $\tau_{L'_i a_{i+1} b_i} = 0$ and $\tau_{L'_i e'_i a_{i+1}} \leq 1$. Then by induction hypothesis, L'_i contains a D_i-Tutte path Q_i between b_i and a_{i+1} such that $e'_i \in E(Q_i)$ and $\beta_{L'_i}(Q_i) \leq (|Q_i| - 6)/2 + 3/2 = (|Q_i|-1)/2 - 1$. ∎

We now form the path P by taking the union of P_H, P_i for $i \in [m]$, and Q_i for $i \in [m-1]$. Clearly, P is between u and v and contains e.

It is easy to see that if B is a P-bridge of G then B is a P_H-bridge of H, or a $(P_i \cup (J_i \cap P_H)$-bridge of J_i, for some $J_i \in B_i$ or a Q_i-bridge of some L_i. Thus, P is a C-Tutte path in G between u and v containing e.

If we extend P_H from v' to v through $J_1, L_1, J_2, L_2, \ldots, J_{m-1}, L_{m-1}, J_m$ in order, we see that

- J_1 and H double count v';

- J_m and $H \cup (J_1 \cup L_1) \cup \ldots \cup (J_{m-1} \cup L_{m-1})$ double count a_m;
- for $1 < i < m$, P_i and $P_H \cup (P_1 \cup Q_1) \cup \ldots \cup (P_{i-1} \cup Q_{i-1})$ double count a_i.
- for $1 < i < m$, Q_i and $P_H \cup (P_1 \cup Q_1) \cup \ldots \cup (P_{i-1} \cup Q_{i-1}) \cup P_i$ double count b_i.

Note that for each $J_i \in \mathcal{J}_2$, the P_H-bridge of H contained in J_i does not contribute to the count of $\beta_G(P)$. We calculate $\beta_G(P)$ as follows.

$$\beta_G(P) = \beta_H(P_H) + \beta_{J_1}(P_1) + \beta_{J_m}(P_m + w) + \sum_{J_i \in \mathcal{J}_1} \beta_{J_i}(P_i \cup (J_i \cap P_H)) + \sum_{J_i \in \mathcal{J}_2} (\beta_{J_i}(P_i \cup (J_i \cap P_H)) - 1) + \sum_{i=1}^{m-1} \beta_{L_i}(Q_i).$$

Suppose eCv is not good. Then $\tau_{Gev} = 1$.Thus, by Claims 2.2.3, 2.2.4, 2.2.5, 2.2.6, 2.2.7, and 2.2.8, we have

$$\begin{aligned}
\beta_G(P) &\leq ((|P_H| - 6)/2 + \tau_{Gue} + 3/2) + ((|P_1| - 1)/2 - 1/2) + ((|P_m| - 1)/2 + \tau_{Gvu}) \\
&\quad + \sum_{J_i \in \mathcal{J}_1 \cup \mathcal{J}_2} (|P_i| - 1)/2 + \sum_{L_i \in \mathcal{L}} \max\{0, (|Q_i| - 1)/2 - 1\} \\
&\leq (|P| - 6)/2 + 1 + \tau_{Gvu} + \tau_{Gue} \\
&= (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.
\end{aligned}$$

So suppose eCv is good. Let $\mathcal{J}_1' := \{J_i \in \mathcal{J}_1$ or $i = m : |a_iCb_i| \geq 3\}$, $\mathcal{J}_1'' := \{J_i \in \mathcal{J}_1$ or $i = m : |a_iCb_i| = 2\}$, $\mathcal{J}_2' := \{J_i \in \mathcal{J}_2 : a_i \neq b_i\}$, $\mathcal{J}_2'' := \{J_i \in \mathcal{J}_2 : a_i = b_i\}$, $\mathcal{L}' := \{L_i \in \mathcal{L} : |b_iCa_{i+1}| \geq 3\}$, and $\mathcal{L}'' := \{L_i \in \mathcal{L} : |b_iCa_{i+1}| = 2\}$. Note then by Claims 2.2.3, 2.2.4, 2.2.5, 2.2.6, 2.2.7, and 2.2.8 we have,

$$\beta_G(P) \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + 1 - |\mathcal{J}_1'| - |\mathcal{J}_1''|/2 - |\mathcal{J}_2'| - |\mathcal{L}'| - |\mathcal{L}''|/2.$$

We may assume $|\mathcal{J}_1'| = |\mathcal{J}_2'| = |\mathcal{L}'| = 0$ as otherwise (1.1) holds. If $|eCv| = 2$, then $\tau_{Gev} = 1/2$ and if $|eCv| \geq 3$, then $\tau_{Gev} = 0$. As $|\mathcal{J}_1''| + |\mathcal{L}''| = |eCv| - 1$ (follows from

$|\mathcal{J}_1'| = |\mathcal{J}_2'| = |\mathcal{L}'| = 0$), (1.1) holds and Theorem 1.2.1 follows. $\square$

We now apply Theorem 1.2.1 to recover the original technical theorem of Wigal and Yu [68]. A challenge to their theorem was accounting for the vertices in the bridges of the path. Care had to be taken to avoid double counting these vertices, as the path may need to be extended through a bridge in a previous iteration of the induction. To handle this, a contraction strategy is employed.

Corollary 2.2.9. *[68] Let $n \geq 3$ be an integer, let (G, C) be a circuit graph on n vertices, let $u, v \in V(C)$ be distinct, and let $e \in E(C)$, such that u, e, v occur on C in clockwise order. Then G has a C-Tutte path between u and v such that $e \in E(P)$ and*

$$\beta(P) \leq (n-6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3.$$

Proof. First suppose $e = uv$. Letting $P := uv$, we have that $\tau_{Gvu} = 1, \tau_{Gue} = 1$, and $\tau_{Gev} = 1$. As $\beta_G(P) = 1$ and $n \geq 3$, the inequality holds. Now suppose $e \neq uv$ and $|G| = 3$. By symmetry, we may assume u is not incident with e. Let $P := C - uv$. As $\beta_G(P) = 0$, $\tau_{Gvu} = 0$, $\tau_{Gue} = 1/2$, and $\tau_{Gev} = 1$, the inequality holds. Thus we may assume $e \neq uv$ and $|G| > 3$ and we proceed with induction on n.

Claim 2.2.10. *If (G_1, G_2) is a 2-separation in G such that $u, v \in V(G_1)$, $e \in E(G_1)$, and $V(G_1 \cap G_2) \subseteq V(C)$, then $|G_2| = 3$.*

Proof. For otherwise, let (G_1, G_2) be a 2-separation in G such that $u, v \in V(G_1)$, $e \in E(G_1)$, $V(G_1 \cap G_2) \subseteq V(C)$, and $|G_2| > 3$. Let $V(G_1 \cap G_2) = \{x, y\}$ such that $yCx \subseteq G_1$ and $xCy \subseteq G_2$. Let $G_1' := G_1 + \{t, tx, ty\}$, where t is a new vertex and $C_1 := yCxty$. Then (G_1', C_1) is a circuit graph. We apply induction on G_1 to find a Tutte path P_1 in G_1 between u and v such that $e \in E(P_1)$ and

$$\beta_{G_1'}(P_1) \leq (|G_1'| - 6)/3 + (2\tau_{G_1'vu} + 2\tau_{G_1'ue} + 2\tau_{G_1'ev})/3$$

If $t \notin V(P_1)$, then let $P := P_1$. As $|G'_1| \leq n$, $\tau_{G'_1vu} = \tau_{Gvu}, \tau_{G'_1ue} = \tau_{Gue}$, and $\tau_{G'_1ev} = \tau_{Gev}$, P is the desired Tutte path.

So suppose $t \in V(P_1)$. Let $C_2 := xCy + yx$ and $G'_2 := G_2 + xy$. Note then (G'_2, C_2) is a circuit graph. As $|xCy| \geq 3$, choose edge $e' \in E(xCy)$ such that $\tau_{G'_2xe} \leq 1/2$. As $\tau_{G'_2yx} = 0$, by induction, G'_2 has a C_2-Tutte path P_2 between x and y such that $e' \in E(P_2)$ and

$$\begin{aligned}\beta_{G'_2}(P_2) &\leq (|G'_2| - 6)/3 + (2\tau_{G'_2yx} + 2\tau_{G'_2xe'} + 2\tau_{G'_2e'y})/3 \\ &= (|G'_2| - 3)/3\end{aligned}$$

Let $P := (P_1 - t) \cup P_2$. As $|G'_1| + |G'_2| = n + 3$ we have

$$\begin{aligned}\beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\ &\leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3 + (|G'_2| - 3)/3 \\ &= (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3.\end{aligned}$$

∎

Claim 2.2.11. *If there is a 3-separation (G_1, G_2) in G such that $C \subseteq G_1$, then $|G_2| = 4$.*

Proof. Assume otherwise, and let (G_1, G_2) be such a separation that minimizes $|V(G_2)| + |E(G_2)|$. Let $V(G_1 \cap G_2) = \{x, y, z\}$.

Let G'_1 be the graph with $V(G'_1) = V(G_1) \cup \{t\}$ where t is a new vertex in the face of G_1 containing G_2, and $E(G'_1) = E(G_1) \cup \{xt, yt, zt\}$. Note that (G'_1, C) is a circuit graph; so by induction there exists a C-Tutte path P_1 between u and v such that $e \in E(P_1)$ and

$$\beta_{G'_1}(P_1) \leq (|G'_1| - 6)/3 + (2\tau_{G'_1vu} + 2\tau_{G'_1ue} + 2\tau_{G'_1ev})/3.$$

If $t \notin V(P_1)$, we let $P := P_1$. As $\tau_{G'_1vu} = \tau_{Gvu}, \tau_{G'_1ue} = \tau_{Gue}$, $\tau_{G'_1ev} = \tau_{G'_1ev}$, and

$|G'_1| < n$, we have that P is the desired Tutte path.

Thus we may assume $t \in V(P_1)$. Without loss of generality we may assume $xt, yt \in E(P_1)$ and x, z, y appear on C_2 in clockwise order.

Suppose $z \in V(P)$. Let $G'_2 := G_2 + \{xz, yz\}$ and C_2 be the outer cycle of G'_2 such that $xzy \subseteq C_2$. Then (G'_2, C_2) is a circuit graph. By induction, as xC_2z and yC_2x are both good and $|yC_2x| \geq 3$, there exists a Tutte path P_2 in G'_2 between x and z such that $yz \in E(P_2)$ and

$$\beta_{G'_2}(P_2) \leq (|G'_2| - 4)/3.$$

Let $P = (P_1 - t) \cup (P_2 - z)$. As $|G'_1| + |G'_2| = |G| + 4$, we have that

$$\begin{aligned}
\beta_G(P) &= \beta_{G'_1}(P_1) + \beta_{G'_2}(P_2) \\
&\leq (|G'_1| - 6)/3 + (2\tau_{G'_1 vu} + 2\tau_{G'_1 ue} + 2\tau_{G'_1 ev})/3 + (|G'_2| - 4)/3 \\
&= (n - 6)/3 + (2\tau_{G'_1 vu} + 2\tau_{G'_1 ue} + 2\tau_{G'_1 ev})/3.
\end{aligned}$$

Now assume $z \notin V(P_1)$. Let G'_2 be the block of $G_2 + xy - \{xz, yz\}$ containing x and y and C_2 be its outer cycle. If $z \in V(G'_2)$ we let $z' := z$, otherwise we let $z' \in V(G_2)$ be the cut vertex of $G_2 + xy$ separating z from x and y. We may assume x, z', y appear on C_2 in clockwise order.

First suppose $z' \neq z$. By Lemma 2.1.4, G'_2 has a C_2-Tutte path P_2 between x and y such that $z' \in V(P_2)$ and

$$\beta_{G'_2}(P_2) \leq (|P_2| - 3)/2.$$

Note $\beta_{G'_2}(P_2) + |P_2| = |G'_2|$ as otherwise we could find a 3-cut contradicting the choice of

(G_1, G_2), the minimality of $|V(G_2)| + |E(G_2)|$. Thus

$$|G_2'| = |P_2| + \beta_{G_2'}(P_2) \leq (3|P_2| - 3)/2.$$

Thus $(2|G_2'|+3)/3 \leq |P_2|$ and $\beta_{G_2'}(P_2) = |G_2'|-|P_2| \leq (|G_2'|-3)/3$. Let $P = (P_1-t)\cup P_2$. As $|G_1'| + |G_2'| = |G| + 3$ (as $z \neq z'$), we have

$$\begin{aligned}
\beta_G(P) &= \beta_{G_1'}(P_1) + \beta_{G_2'}(P_2) \\
&\leq (|G_1'| - 6)/3 + (2\tau_{G_1'vu} + 2\tau_{G_1'ue} + 2\tau_{G_1'ev})/3 + (|G_2'| - 3)/3 \\
&= (n - 6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3.
\end{aligned}$$

Now suppose $z = z'$. As $xz, yz \notin E(G_2')$, there is a choice of $e' \in E(C_2)$ such that e' contains z' and $\tau_{G_2'e'y} = 0$. By Theorem 1.2.1, G_2' has a C_2-Tutte path P_2 between x and y such that $e' \in E(P_2)$ and

$$\beta_{G_2'}(P_2) \leq (|P_2| - 4)/2.$$

Again, by our choice of (G_1, G_2), we have $\beta_{G_2'}(P_2)+|P_2| = |G_2'|$ and $|G_2'| \leq (3|P_2|-4)/2$. In particular $(2|G_2'| + 4)/3 \leq |P_2|$ and $\beta_{G_2'}(P_2) = |G_2'| - |P_2| \leq (|G_2'| - 4)/3$. Letting $P = (P_1 - t) \cup P_2$, as $|G_1'| + |G_2'| = |G| + 4$, we have

$$\begin{aligned}
\beta_G(P) &= \beta_{G_1'}(P_1) + \beta_{G_2'}(P_2) \\
&\leq (|G_1'| - 6)/3 + (2\tau_{G_1'vu} + 2\tau_{G_1'ue} + 2\tau_{G_1'ev})/3 + (|G_2'| - 4)/3 \\
&= (n - 6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3.
\end{aligned}$$

∎

Let (G, C) be a circuit graph on $n \geq 3$ vertices. By Theorem 1.2.1, there exists a

C-Tutte path P between u and v such that

$$\beta_G(P) \leq (|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.$$

By Claims 2.2.10 and 2.2.11 we have $n = |P| + \beta_G(P)$. It follows,

$$n = |P| + \beta_G(P) \leq (3|P| - 6)/2 + \tau_{Gvu} + \tau_{Gue} + \tau_{Gev}.$$

Solving for $|P|$ we have

$$(2n + 6)/3 - 2(\tau_{Gvu} + \tau_{Gue} + \tau_{Gev})/3 \leq |P|.$$

Thus we have

$$\beta_G(P) = n - |P| \leq (n - 6)/3 + (2\tau_{Gvu} + 2\tau_{Gue} + 2\tau_{Gev})/3.$$

□

2.3 Essentially 4-connected Planar Graphs

We now apply Corollary 2.2.9 to obtain sharp circumference bounds for essentially 4-connected graphs.

Theorem 2.3.1. *[68] Let $n \geq 6$ be an integer and let G be any essentially 4-connected n-vertex planar graph. Then the circumference of G is at least $\lceil (2n + 6)/3 \rceil$.*

Proof. First suppose G is 4-connected. Fix a planar drawing of G and let T be the outer cycle of G. Let $uv, e \in E(T)$ be distinct. By applying Corollary 2.2.9, G has a T-Tutte path P between u and v such that $e \in E(P)$. As G is 4-connected, $\beta_G(P) = 0$, so P is in fact a Hamiltonian path. Hence, $P + uv$ is a Hamiltonian cycle in G and has length n, which is at least $(2n + 6)/3$ (as $n \geq 6$).

Hence, we may assume G is not 4-connected. Then, since G is essentially 4-connected, there exists $x \in V(G)$ such that x has degree 3 in G. So let $N_G(x) = \{u, v, w\}$ and let $H := G - x$ and assume that H is a plane graph with u, v, w on the outer cycle C of H in counter clockwise order. Note that (H, C) is a circuit graph.

Suppose two of $|uCw|, |wCv|, |vCu|$ is at least 3. Without loss of generality, we may assume that $|uCw| \geq 3$ and $|wCv| \geq 3$. Let $e \in E(uCw)$ be incident with w. Then $\tau_{Hvu} = 0, \tau_{Hue} \leq 1/2$, and $\tau_{Hev} = 0$. Hence by prior Theorem, H has a C-Tutte path between u and v such that $e \in E(P)$ and $\beta_H(P) \leq (n-7)/2 + 1/3 = (n-6)/3$. Thus, $Q := P \cup uxv$ is a Tutte cycle in G such that $\beta_G(Q) \leq (n-6)/3$. Since G is essentially 4-connected, every Q-bridges is a $K_{1,3}$. Hence $|Q| \geq n - (n-6)/3 = (2n+6)/3$.

So we may assume that $|wCv| = |vCu| = 2$. Consider the plane graph $K := H - wv$ whose outer cycle D contains vCw. Since G is essentially 4-connected, K is 2-connected; so (K, D) is a circuit graph. We can choose $e \in E(wDv)$ incident with w. Now $\tau_{Kvu} = 0$, $\tau_{Kue} \leq 1/2$, and $\tau_{Kev} \leq 1/2$.

If $\tau_{Kue} = 0$ or $\tau_{Kev} = 0$, then by Corollary 2.2.9, K has a D-Tutte path P between u and v such that $e \in E(P)$ and $\beta_K(P) \leq (n-7)/3 + 1/3 = (n-6)/3$. Thus $Q := P \cup uxv$ is a cycle in G with $|Q| \geq n - (n-6)/3 = (2n+6)/3$.

So assume $\tau_{Kue} = \tau_{Kev} = 1/2$ and, hence, $|wDv| = 3$ and $|uDw| = 2$. Since $n \geq 6$ and G is essentially 4-connccted, one of $\{v, w\}$ has a neighbor inside D, say w by symmetry. Now consider the plane graph $J := H - uw$, which is 2-connected as G is essentially 4-connected. Let F denote the outer cycle of J, which contains $\{u, v, w\}$. Clearly, (J, F) is a circuit graph. Choose $f \in E(uFw)$ incident with w. Then $\tau_{Juf} \leq 1/2$, and $\tau_{Jfv} = 0$, and $\tau_{Jvu} = 0$. Hence, by Corollary 2.2.9, J has an F-Tutte path between u and v such that $f \in E(P)$ and $\beta_J(P) \leq (n-7)/3 + 1/3 = (n-6)/3$. Thus $Q := P \cup uxv$ is a cycle in G with $|Q| \geq n - (n-6)/3 = (2n+6)/3$.

$\square$

Note that we need $n \geq 6$ in Theorem 2.3.1, however, when $n \geq 5$ the graph G is

Hamiltonian. The bound in Theorem 2.3.1 is best possible in the following sense. Take a 4-connected triangulation T on k vertices, and inside each face of T add a new vertex and three edges from that new vertex to the three vertices in the boundary of that face. The resulting, graph, say G, has $n := 3k - 4$ vertices. Now take an arbitrary cycle C in G. For each $x \in V(C)$ with degree three in G, we delete x from C and add the edge of G between the two neighbors of x in C. The results in a cycle in T, say D. Clearly, $|D| \leq k$; which implies $|C| \leq 2k$. Hence, the circumference of G is at most $2k = (2n+4)/3 = \lceil (2n+6)/3 \rceil$.

CHAPTER 3

EVEN COVERS

In this chapter, we provide results on even covers in subcubic graphs and apply them to the travelling salesperson problem. This chapter is joint work with Youngho Yoo and Xingxing Yu [66].

3.1 Subcubic Chains

In order to help with induction, we consider even covers which contain or avoid a specified edge. Let G be a graph and let $e \in E(G)$. We write $\mathcal{E}(G, e)$ to denote the set of even covers of G containing e, and $\widehat{\mathcal{E}}(G, e)$ to denote the set of even covers of G not containing e. Define

$$\text{exc}(G, e) := \min_{F \in \mathcal{E}(G,e)} \text{exc}(F) - 2$$

$$\widehat{\text{exc}}(G, e) := \min_{F \in \widehat{\mathcal{E}}(G,e)} \text{exc}(F)$$

Clearly, we have $\text{exc}(G) = \min\{\text{exc}(G, e) + 2, \widehat{\text{exc}}(G, e)\}$ for any edge $e \in E(G)$. The "-2" in the definition of $\text{exc}(G, e)$ leads to a natural interpretation of the quantities $\delta(G, e)$ and $\widehat{\delta}(G, e)$ defined below, and also results in simpler calculations as it accounts for the fact that the cycle C of F containing e will often only be used as a path $C - e$ as part of a larger cycle (see Propositions 3.1.1 and 3.1.2).

To prove (1.3), it will be convenient to define the following parameters for a graph G

and an edge $e \in E(G)$:

$$\delta(G, e) := \text{exc}(G, e) - \frac{n(G) + n_2(G)}{4},$$
$$\widehat{\delta}(G, e) := \widehat{\text{exc}}(G, e) - \frac{n(G) + n_2(G)}{4}.$$

Note that if every vertex of G has degree 2 or 3 (for instance, if G is subcubic and 2-connected), then $\delta(G, e)$ and $\widehat{\delta}(G, e)$ are always half-integral since $n(G) + n_2(G) = (n(G) - n_2(G)) + 2n_2(G)$ where $(n(G) - n_2(G))$ is the number of vertices of odd degree in G, which is always even.

A *subcubic chain* C is a simple connected subcubic graph, written as an alternating sequence $C = xe_0B_1e_1B_2 \dots B_ke_ky$ for some nonnegative integer k, satisfying the following properties (see Figure 3.1):

- $\{e_0, \dots, e_k\}$ is the set of cut-edges of C,
- $\{B_0, B_1, \dots, B_k, B_{k+1}\}$ is the set of connected components of $C - \{e_0, \dots, e_k\}$, where $V(B_0) = \{x\}$ and $V(B_{k+1}) = \{y\}$,
- B_i is either a single vertex or 2-connected for all $i \in [k]$, and
- each e_i has one endpoint in B_i and one endpoint in B_{i+1} for all $i = 0, \dots, k$.

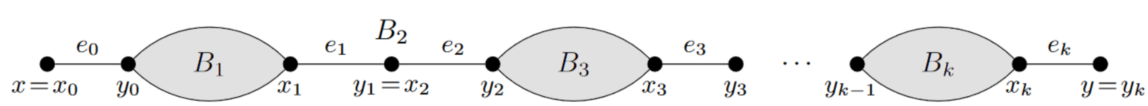

Figure 3.1: A subcubic chain

We say that C has *end points* x, y and has *end edges* e_0 and e_k. A subcubic chain is *trivial* if $k = 0$ (that is, C is an edge xy), and *nontrivial* otherwise.

Let $C = xe_0B_1e_1B_2 \dots B_ke_ky$ be a nontrivial subcubic chain. For $i \in [k]$, let x_i denote the endpoint of e_{i-1} in B_i and let y_i denote the endpoint of e_i in B_i. (Note that $x_i \neq y_i$

when $n(B_i) \neq 1$, as C is subcubic.) We define $\overline{B_i} = B_i + \overline{e_i}$ where $\overline{e_i} = x_iy_i$, and $\overline{C} = C - \{x, y\} + e_C$ where $e_C = x_1y_k$. We call each $(\overline{B_i}, \overline{e_i})$ a *chain-block* of C, and $\overline{C}$ the *closure* of C. Note that the closure of a nontrivial subcubic chain C is a subcubic graph with no cut-vertex such that $\overline{C} - e_C$ is simple. If C is a trivial subcubic chain, we define $\mathrm{exc}(\overline{C}, e_C) = \widehat{\mathrm{exc}}(\overline{C}, e_C) = \delta(\overline{C}, e_C) = \widehat{\delta}(\overline{C}, e_C) = 0$.

Proposition 3.1.1. *Let $C = xe_0B_1e_1B_2 \ldots B_ke_ky$ be a subcubic chain, and let $\{(\overline{B_i}, \overline{e_i}) : i \in [k]\}$ denote the chain-blocks of C. Then*

- $\mathrm{exc}(\overline{C}, e_C) = \sum_{i=1}^{k} \mathrm{exc}(\overline{B_i}, \overline{e_i})$*,*
- $\widehat{\mathrm{exc}}(\overline{C}, e_C) = \sum_{i=1}^{k} \widehat{\mathrm{exc}}(\overline{B_i}, \overline{e_i})$*,*
- $\delta(\overline{C}, e_C) = \sum_{i=1}^{k} \delta(\overline{B_i}, \overline{e_i})$*, and*
- $\widehat{\delta}(\overline{C}, e_C) = \sum_{i=1}^{k} \widehat{\delta}(\overline{B_i}, \overline{e_i})$*.*

Proof. If C is trivial then the proposition is true by definition (an empty sum is defined to be 0), so we may assume that C is nontrivial. Note that a cycle in $\overline{C}$ contains e_C if and only if it contains all of $e_1, \ldots, e_{k-1}$. This gives a natural bijective correspondence between even covers $F \in \mathcal{E}(\overline{C}, e_C)$ and tuples of even covers $(F_1, \ldots, F_k)$ where $F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})$ for each $i \in [k]$. Indeed, this correspondence is obtained by "splitting" the cycle D of F containing e_C into k cycles, $(D \cap \overline{B_i}) + \overline{e_i}$ for $i \in [k]$. With this correspondence, we have $\mathrm{exc}(F) = 2 + \sum_{i=1}^{k}(\mathrm{exc}(F_i) - 2)$. Hence,

$$\begin{aligned}
\mathrm{exc}(\overline{C}, e_C) &= \min_{F \in \mathcal{E}(\overline{C}, e_C)} \mathrm{exc}(F) - 2 \\
&= \sum_{i=1}^{k} \min_{F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})} (\mathrm{exc}(F_i) - 2) \\
&= \sum_{i=1}^{k} \mathrm{exc}(\overline{B_i}, \overline{e_i}).
\end{aligned}$$

Since $n(\overline{C}) = \sum_{i=1}^{k} n(\overline{B_i})$ and $n_2(\overline{C}) = \sum_{i=1}^{k} n_2(\overline{B_i})$, this also implies $\delta(\overline{C}, e_C) = \sum_{i=1}^{k} \delta(\overline{B_i}, \overline{e_i})$.

Similarly, there is a natural bijective correspondence between even covers $F \in \widehat{\mathcal{E}}(\overline{C}, e_C)$ and tuples $(F_1, \ldots, F_k)$ where $F_i \in \widehat{\mathcal{E}}(\overline{B_i}, \overline{e_i})$ for each $i \in [k]$. That is, F_i is the restriction of F on B_i for all $i \in [k]$. Moreover, $\text{exc}(F) = \sum_{i=1}^k \text{exc}(F_i)$. Hence,

$$\begin{aligned}
\widehat{\text{exc}}(\overline{C}, e_C) &= \min_{F \in \widehat{\mathcal{E}}(\overline{C}, e_C)} \text{exc}(F) \\
&= \sum_{i=1}^{k} \min_{F_i \in \widehat{\mathcal{E}}(\overline{B_i}, \overline{e_i})} \text{exc}(F_i) \\
&= \sum_{i=1}^{k} \widehat{\text{exc}}(\overline{B_i}, \overline{e_i}).
\end{aligned}$$

This similarly gives $\widehat{\delta}(\overline{C}, e_C) = \sum_{i=1}^k \widehat{\delta}(\overline{B_i}, \overline{e_i})$. □

The parameters $\delta(\overline{C}, e_C)$ and $\widehat{\delta}(\overline{C}, e_C)$ can be interpreted as the "difference" in the δ or $\widehat{\delta}$ of the overall graph G made by the presence of the subcubic chain C compared to a trivial chain (a single edge). This is formalized in the next proposition.

Let G be a graph containing a nontrivial subcubic chain $C = xe_0B_1 \ldots B_ke_ky$ such that $C - \{x, y\}$ is a connected component of $G - \{e_0, e_k\}$. In this case, we say that C is a *subcubic chain of* G. If C is a subcubic chain of G, we write G/C to denote the graph obtained by suppressing $V(C) \setminus \{x, y\}$, and write $e_{G/C}$ to denote the resulting edge. We say that G/C is obtained from G by *suppressing* C. A cycle in G containing the edge e_0 (hence all of $\{e_0, \ldots, e_k\}$) is said to be a *cycle through* C, and an *even cover through* C is an even cover of G containing a cycle through C.

Proposition 3.1.2. *Let C be a subcubic chain of a graph G, and let e be a cut-edge of C. Then $\delta(G, e) = \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C)$ and $\widehat{\delta}(G, e) = \widehat{\delta}(G/C, e_{G/C}) + \widehat{\delta}(\overline{C}, e_C)$.*

Proof. Given an even cover $F \in \mathcal{E}(G, e)$, e is contained in some cycle D in F. By splitting D into two cycles $(D \cap G/C) + e_{G/C}$ and $(D \cap C) + e_C$, we obtain from F two even covers $F' \in \mathcal{E}(G/C, e_{G/C})$ and $F_C \in \mathcal{E}(\overline{C}, e_C)$ satisfying $\text{exc}(F) = \text{exc}(F') + \text{exc}(F_C) - 2$. This bijective correspondence gives

$$\begin{aligned}
\mathrm{exc}(G,e) &= \min_{F\in\mathcal{E}(G,e)} \mathrm{exc}(F)-2\\
&= \min_{F'\in\mathcal{E}(G/C,e_{G/C})}(\mathrm{exc}(F')-2)+\min_{F_C\in\mathcal{E}(\overline{C},e_C)}(\mathrm{exc}(F_C)-2)\\
&= \mathrm{exc}(G/C,e_{G/C})+\mathrm{exc}(\overline{C},e_C).
\end{aligned}$$

Similarly, for any even cover $F\in\widehat{\mathcal{E}}(G,e)$, its restriction on G/C is in $\widehat{\mathcal{E}}(G/C,e_{G/C})$ and its restriction on $\overline{C}$ is in $\widehat{\mathcal{E}}(\overline{C},e_C)$; and we have $\widehat{\mathrm{exc}}(G,e)=\widehat{\mathrm{exc}}(G/C,e_{G/C})+\widehat{\mathrm{exc}}(\overline{C},e_C)$.

Since $n(G)=n(G/C)+n(\overline{C})$ and $n_2(G)=n_2(G/C)+n_2(\overline{C})$, the proposition follows from the definitions of δ and $\widehat{\delta}$. □

We will show in Theorem 3.1.4 that $\delta(G,e)+\widehat{\delta}(G,e)\le 0$ for every 2-connected subcubic graph G and every edge $e\in E(G)$ for which $G-e$ is simple. If $\delta(G,e)+\widehat{\delta}(G,e)=0$, then we say that (G,e) is *tight*. A subcubic chain C is *tight* if its closure $(\overline{C},e_C)$ is tight.

The next proposition states that a subcubic chain is tight if and only if all of its chain-blocks are tight.

Proposition 3.1.3. *Let $C=xe_0B_1e_1B_2\dots B_ke_ky$ be a subcubic chain, and assume $\delta(\overline{B_i},\overline{e_i})+\widehat{\delta}(\overline{B_i},\overline{e_i})\le 0$ for all i. Then $\delta(\overline{C},e_C)+\widehat{\delta}(\overline{C},e_C)\le 0$, with equality if and only if $\delta(\overline{B_i},\overline{e_i})+\widehat{\delta}(\overline{B_i},\overline{e_i})=0$ for all $i\in[k]$.*

Proof. Since $\delta(\overline{B_i},\overline{e_i})+\widehat{\delta}(\overline{B_i},\overline{e_i})\le 0$ for all i, we have by Proposition 3.1.1,

$$\delta(\overline{C},e_C)=\sum_{j=1}^{k}\delta(\overline{B_i},\overline{e_i})\le\sum_{j=1}^{k}(-\widehat{\delta}(\overline{B_i},\overline{e_i}))=-\widehat{\delta}(\overline{C},e_C).$$

Hence, $\delta(\overline{C},e_C)+\widehat{\delta}(\overline{C},e_C)\le 0$, with equality if and only if $\delta(\overline{B_i},\overline{e_i})+\widehat{\delta}(\overline{B_i},\overline{e_i})=0$ for all i. □

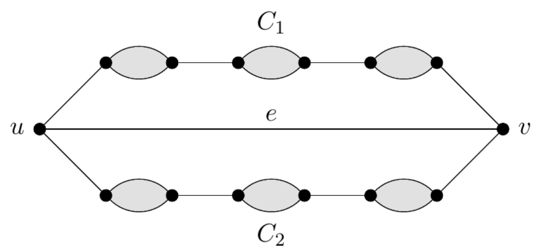

Figure 3.2: A rooted θ-chain

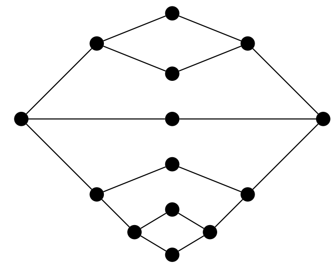

Figure 3.3: A minimal θ-chain

We say that a subcubic chain C is *minimal* if it is tight and $\delta(\overline{C}, e_C) = -\frac{1}{2}$, and that C is *near-minimal* if it is tight and $\delta(\overline{C}, e_C) \in \{-\frac{1}{2}, -1\}$. Two subcubic chains C_1 and C_2 are *balanced* if $\delta(\overline{C_1}, e_{C_1}) = \delta(\overline{C_2}, e_{C_2})$.

A *θ-chain* is a graph G that is the union of three internally disjoint subcubic chains C_1, C_2, C_3 with common endpoints. We call C_1, C_2, C_3 the *chains of G*. Note that the choices of the three chains C_1, C_2, C_3 may not be unique (consider the graph obtained from two disjoint 4-cycles by adding two edges joining them so that the endpoints of the two edges are nonadjacent in each 4-cycle). A *rooted θ-chain* is a pair (G, e) where G is a graph and $e = uv \in E(G)$ such that $G - e$ is the union of two internally disjoint subcubic chains C_1, C_2 with common endpoints $\{u, v\}$. We call C_1, C_2 the *chains of* (G, e). See Figure 3.2.

A (rooted) θ-chain is *balanced* if all pairs of its chains are balanced, *tight* if the closures of its chains are all tight, and *(near) minimal* if all of its chains are (near) minimal. Note that a (near) minimal (rooted) θ-chain is also balanced and tight by definition. See Figure 3.3.

We can now state our main result, which immediately implies (1.3). For inductive purposes, we allow the graph G to be a loop e on a single vertex and we also allow one edge of $G - e$ to be parallel to e. In all cases however, $G - e$ is a simple subcubic graph.

Theorem 3.1.4. *Let G be a 2-connected subcubic graph and let $e = uv$ be an edge of G such that $G - e$ is simple. Then the following statements hold:*

(T1) $\delta(G, e) \leq -\frac{1}{2}$*, with equality if and only if either G is a loop or (G, e) is a balanced tight rooted θ-chain.*

(T2) *If $G - e$ is 2-connected, then $\widehat{\delta}(G, e) \leq \frac{3}{2}$, with equality if and only if $G - e$ is a minimal θ-chain.*

(T3) *If $\delta(G, e) = -1$, then either*

(a) $G \cong K_4$*, or*

(b) e *has a parallel edge, and suppressing $\{u, v\}$ to an edge e' results in a graph G' such that either G' is a loop or (G', e') is a near-minimal rooted θ-chain, or*

(c) there exists $e' \in E(G)$ such that $\{e, e'\}$ is a 2-edge-cut in G, and suppressing either subcubic chain C of G with end edges e, e' yields either a loop or a balanced tight rooted θ-chain $(G/C, e_{G/C})$, or

(d) (G, e) *is a rooted θ-chain such that* $\min_{i \in [2]} \Big(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}}) \Big) = -\frac{1}{2}$.

(T4) $\delta(G, e) + \widehat{\delta}(G, e) \leq 0$.

One immediate consequence of Theorem 3.1.4 is that if C is a subcubic chain, then $\delta(\overline{C}, e_C) \leq -\frac{1}{2}$ unless C is trivial, in which case $\delta(\overline{C}, e_C) = 0$ by definition. In particular, $\delta(G, e) \leq -\frac{1}{2}$ for every nonempty 2-connected subcubic graph G and $e \in E(G)$ such that $G - e$ is simple. Hence, if C is a minimal subcubic chain, then by Proposition 3.1.1, it has exactly one chain-block $(\overline{B}, \overline{e_B})$, and this chain-block satisfies $\delta(\overline{B}, \overline{e_B}) = -\frac{1}{2}$.

3.2 Properties of θ-chains

In this section, we derive useful properties of balanced, tight, or minimal θ-chains assuming Theorem 3.1.4 for smaller graphs. We begin by proving statements **(T1)** and **(T3)** of Theorem 3.1.4, assuming Theorem 3.1.4 for smaller graphs, for the special case where (G, e) is a rooted θ-chain (equivalently, G is simple and $\{u, v\}$ forms a cut in G). The proof is a relatively straightforward but illustrative demonstration of our techniques.

Lemma 3.2.1. *Let (G, e) be a simple rooted θ-chain, and let C_1, C_2 denote the two chains of (G, e). Assume that Theorem 3.1.4 holds for graphs with fewer than $n(G)$ vertices. Then*

(i) $\delta(G, e) = -\frac{1}{2} + \min_{i\in[2]} \left(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}})\right) \leq -\frac{1}{2}$*, with equality if and only if (G, e) is a balanced tight rooted θ-chain,*

(ii) $\widehat{\delta}(G, e) \leq \frac{3}{2} + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2}) \leq \frac{1}{2}$*,*

(iii) $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{1}{2}, \frac{1}{2})$ *if and only if (G, e) is a minimal rooted θ-chain, and*

(iv) if $\delta(G, e) = -1$ *then* $\min_{i\in[2]} \left(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}})\right) = -\frac{1}{2}$.

Proof. An even cover $F \in \mathcal{E}(G, e)$ corresponds to a pair (F_1, F_2) where $F_i \in \mathcal{E}(\overline{C_i})$ for each $i \in [2]$ and $F_i \in \mathcal{E}(\overline{C_i}, e_{C_i})$ for exactly one $i \in [2]$. This correspondence gives $\text{exc}(F) = \text{exc}(F_1) + \text{exc}(F_2)$. Since $n(G) = n(\overline{C_1}) + n(\overline{C_2}) + 2$ and $n_2(G) = n_2(\overline{C_1}) + n_2(\overline{C_2})$, we have

$$
\begin{aligned}
\text{exc}(G, e) &= \min_{F\in\mathcal{E}(G,e)} \text{exc}(F) - 2 \\
&= \min_{i\in[2]} \left(\min_{F_i\in\mathcal{E}(\overline{C_i},e_{C_i})} (\text{exc}(F_i) - 2) + \min_{F_{3-i}\in\widehat{\mathcal{E}}(\overline{C_{3-i}},e_{C_{3-i}})} \text{exc}(F_{3-i}) \right) \\
&= \min_{i\in[2]} \left(\text{exc}(\overline{C_i}, e_{C_i}) + \widehat{\text{exc}}(\overline{C_{3-i}}, e_{C_{3-i}})\right) \\
&= \min_{i\in[2]} \left(\frac{n(\overline{C_i}) + n_2(\overline{C_i})}{4} + \delta(\overline{C_i}, e_{C_i}) + \frac{n(\overline{C_{3-i}}) + n_2(\overline{C_{3-i}})}{4} + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}}) \right) \\
&= \min_{i\in[2]} \left(\frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \delta(\overline{C_i}, e_{C_i}) + \delta(\overline{C_{3-i}}, e_{C_{3-i}}) \right).
\end{aligned}
$$

Therefore,

$$
\delta(G, e) = -\frac{1}{2} + \min_{i\in[2]} \left(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}})\right), \tag{3.1}
$$

whence for $i \in [2]$,

$$\delta(G,e) \le -\frac{1}{2} + \delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_{3-i}}, e_{C_{3-i}}). \tag{3.2}$$

By assumption, Theorem 3.1.4 holds for $(\overline{C_i}, e_{C_i})$; so $\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_i}, e_{C_i}) \le 0$ for each $i \in [2]$. Adding the two inequalities of (3.2) gives

$$2\delta(G,e) \le -1 + \sum_{i\in[2]} \Big(\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_i}, e_{C_i})\Big) \le -1.$$

Hence,

$$\delta(G,e) \le -\frac{1}{2}. \tag{3.3}$$

Moreover, $\delta(G,e) = -\frac{1}{2}$ if and only if all of the above inequalities are tight, which means $(\overline{C_1}, e_{C_1})$ and $(\overline{C_2}, e_{C_2})$ are tight, and

$$0 = \delta(\overline{C_1}, e_{C_1}) + \widehat{\delta}(\overline{C_2}, e_{C_2}) = \delta(\overline{C_1}, e_{C_1}) - \delta(\overline{C_2}, e_{C_2}).$$

In other words, C_1, C_2 are balanced. Together with (3.1) and (3.3), this proves (i).

If $F_i \in \mathcal{E}(\overline{C_i}, e_{C_i})$ for each $i \in [2]$ then, by merging the cycles in F_i containing e_{C_i} for $i \in [2]$, we obtain an even cover $F \in \widehat{\mathcal{E}}(G,e)$ with $\text{exc}(F) = \text{exc}(F_1) + \text{exc}(F_2) - 2$. So

$$\begin{aligned}
\widehat{\text{exc}}(G,e) &\le \min_{F\in\widehat{\mathcal{E}}(G,e)} \text{exc}(F) \\
&\le \min_{F_1\in\mathcal{E}(\overline{C_1}, e_{C_1})} \text{exc}(F_1) + \min_{F_2\in\mathcal{E}(\overline{C_2}, e_{C_2})} (\text{exc}(F_2) - 2) \\
&= (\text{exc}(\overline{C_1}, e_{C_1}) + 2) + \text{exc}(\overline{C_2}, e_{C_2}) \\
&= \frac{n(\overline{C_1}) + n_2(\overline{C_1})}{4} + \delta(\overline{C_1}, e_{C_1}) + \frac{n(\overline{C_2}) + n_2(\overline{C_2})}{4} + \delta(\overline{C_2}, e_{C_2}) + 2 \\
&= \frac{n(G) + n_2(G)}{4} + \frac{3}{2} + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2}).
\end{aligned}$$

Hence,

$$\widehat{\delta}(G,e) \leq \frac{3}{2} + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2}).$$

Since G is simple, each C_i is a nontrivial chain; so $\delta(\overline{C_i}, e_{C_i}) \leq -\frac{1}{2}$ by the assumption that Theorem 3.1.4 holds for $(\overline{C_i}, e_{C_i})$. This gives $\widehat{\delta}(G,e) \leq \frac{1}{2}$ and proves (ii).

To prove (iii), suppose $(\delta(G,e), \widehat{\delta}(G,e)) = (-\frac{1}{2}, \frac{1}{2})$. Then $\delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2}) = -1$ by (ii). Since $\delta(\overline{C_i}, e_{C_i}) \leq -\frac{1}{2}$ for $i \in [2]$ (by assumption), $\delta(\overline{C_i}, e_{C_i}) = -\frac{1}{2}$ for each $i \in [2]$. Moreover, each $(\overline{C_i}, e_{C_i})$ is tight (by (i)), so (G,e) is a minimal rooted θ-chain.

Finally, note that (iv) follows from (i). $\square$

The next lemma says that given a choice of adding an edge uv_1 or uv_2 to a 2-connected subcubic graph Z, the two quantities $\delta(Z+uv_1, uv_1)$ and $\delta(Z+uv_2, uv_2)$ cannot both be large.

Lemma 3.2.2. *Let Z be a 2-connected simple subcubic graph and let u, v_1, v_2 be three distinct vertices of degree 2 in Z. Assume Theorem 3.1.4 holds for graphs with at most $n(Z)$ vertices. Then $\delta(Z+uv_1, uv_1) + \delta(Z+uv_2, uv_2) \leq -2$.*

Proof. By the assumption that Theorem 3.1.4 holds for graphs with at most $n(Z)$ vertices, we have $\delta(Z+uv_i, uv_i) \leq -\frac{1}{2}$ for each $i \in [2]$, with equality if and only if $(Z+uv_i, uv_i)$ is a balanced tight rooted θ-chain. If both $\delta(Z+uv_1, uv_1) \leq -1$ and $\delta(Z+uv_2, uv_2) \leq -1$, then there is nothing to prove. So we may assume by symmetry that $\delta(Z+uv_1, uv_1) = -\frac{1}{2}$; thus $(Z+uv_1, uv_1)$ is a balanced tight rooted θ-chain. Note that it suffices to show that $\delta(Z+uv_2, uv_2) \leq -\frac{3}{2}$.

Let C_1, C_2 denote the two chains of $(Z+uv_1, uv_1)$. Let us assume without loss of generality that $v_2 \in V(C_1)$. Write $C_1 = v_1 e_0 B_1 e_1 B_2 \ldots B_k e_k u$ (where $k \geq 1$) and write its chain-blocks $(\overline{B_i}, \overline{e_i})$ for all $i \in [k]$. Since C_1, C_2 are balanced, we have $\delta(\overline{C_1}, e_{C_1}) = \delta(\overline{C_2}, e_{C_2})$, and since they are both tight, we have $\delta(\overline{C_i}, e_{C_i}) + \widehat{\delta}(\overline{C_i}, e_{C_i}) = 0$ for $i \in [2]$.

So by Proposition 3.1.3 and the assumption that Theorem 3.1.4 holds for each $(\overline{B_i}, \overline{e_i})$, we have

$$\delta(\overline{B_i}, \overline{e_i}) + \widehat{\delta}(\overline{B_i}, \overline{e_i}) = 0 \qquad \text{for all } i \in [k]. \tag{3.4}$$

Let $\ell \in [k]$ be the unique index such that $v_2 \in B_\ell$. (Note ℓ is well defined as Z is subcubic and v_2 has degree 2 in Z.) Let v' denote the vertex of B_ℓ incident with $e_{\ell-1}$.

Then there is an even cover $F \in \mathcal{E}(Z+uv_2, uv_2)$ obtained from a tuple $(F', F_1, \dots, F_k)$ where $F' \in \mathcal{E}(\overline{C_2}, e_{C_2})$, $F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})$ for each $i \in [\ell-1]$, $F_\ell \in \mathcal{E}(B_\ell + v'v_2, v'v_2)$, and $F_j \in \widehat{\mathcal{E}}(\overline{B_j}, \overline{e_j})$ for each $j = \ell+1, \dots, k$. This gives $\operatorname{exc}(F) - 2 = (\operatorname{exc}(F') - 2) + \sum_{i=1}^{\ell}(\operatorname{exc}(F_i) - 2) + \sum_{j=\ell+1}^{k} \operatorname{exc}(F_j)$. Moreover, since $n(B_\ell + v'v_2) = n(\overline{B_\ell})$ and $n_2(B_\ell + v'v_2) = n_2(\overline{B_\ell})$, we have

$$n(Z + uv_2) = 2 + n(\overline{C_2}) + \sum_{i=1}^{\ell-1} n(\overline{B_i}) + n(B_\ell + v'v_2) + \sum_{j=\ell+1}^{k} n(\overline{B_j}),$$
$$n_2(Z + uv_2) = n_2(\overline{C_2}) + \sum_{i=1}^{\ell-1} n_2(\overline{B_i}) + n_2(B_\ell + v'v_2) + \sum_{j=\ell+1}^{k} n_2(\overline{B_j}).$$

This gives

$$\begin{aligned}
\operatorname{exc}(Z + uv_2, uv_2) \le{}& \operatorname{exc}(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \operatorname{exc}(\overline{B_i}, \overline{e_i}) \\
&+ \operatorname{exc}(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^{k} \widehat{\operatorname{exc}}(\overline{B_j}, \overline{e_j}) \\
={}& \frac{n(Z+uv_2) + n_2(Z+uv_2)}{4} - \frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \\
&+ \delta(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}),
\end{aligned}$$

whence

$$\delta(Z + uv_2, uv_2) \le -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) + \delta(B_\ell + v'v_2, v'v_2) + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e}^j).$$

Note that $\widehat{\text{exc}}(\overline{B_\ell}, \overline{e_\ell}) = \widehat{\text{exc}}(B_\ell + v'v_2, v'v_2)$ since both quantities are equal to the minimum excess of an even cover of B_ℓ. This implies $\widehat{\delta}(\overline{B_\ell}, \overline{e_\ell}) = \widehat{\delta}(B_\ell + v'v_2, v'v_2)$. Using (3.4) and that $\delta(B_\ell + v'v_2, v'v_2) + \widehat{\delta}(B_\ell + v'v_2, v'v_2) \leq 0$ as Theorem 3.1.4 holds for $(B_\ell + v'v_2, v'v_2)$ (by assumption), we have

$$
\begin{aligned}
\delta(Z + uv_2, uv_2) &\leq -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \big(-\widehat{\delta}(\overline{B_i}, \overline{e_i})\big) \\
&\quad + \big(-\widehat{\delta}(B_\ell + v'v_2, v'v_2)\big) + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \\
&= -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{i=1}^{\ell-1} \big(-\widehat{\delta}(\overline{B_i}, \overline{e_i})\big) + \big(-\widehat{\delta}(\overline{B_\ell}, \overline{e_\ell})\big) + \sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \\
&= -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \sum_{j=1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) - 2\sum_{j=1}^{\ell} \widehat{\delta}(\overline{B_i}, \overline{e_i}) \\
&= -\frac{1}{2} + \delta(\overline{C_2}, e_{C_2}) + \widehat{\delta}(\overline{C_1}, e_{C_1}) - 2\sum_{j=1}^{\ell} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Proposition 3.1.1)} \\
&= -\frac{1}{2} - 2\sum_{j=1}^{\ell} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \qquad\qquad \text{(as } C_1 \text{ and } C_2 \text{ are balanced and tight)} \\
&\leq -\frac{3}{2},
\end{aligned}
$$

since $-\widehat{\delta}(\overline{B_j}, \overline{e_j}) = \delta(\overline{B_j}, \overline{e_j}) \leq -1/2$ for all $j \in [k]$ by (3.4) and the assumption that Theorem 3.1.4 holds for $(\overline{B_j}, \overline{e_j})$. □

We can now prove the following lemma for θ-chains.

Lemma 3.2.3. *Let G be a subcubic graph with $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected. Assume that Theorem 3.1.4 holds for graphs with fewer than $n(G)$ vertices. Let G_u be the graph obtained from $G - e$ by suppressing u into an edge f_u, and assume that (G_u, f_u) is a rooted θ-chain. Then*

(i) $\widehat{\delta}(G, e) \leq \frac{3}{2}$, with equality if and only if $G - e$ is a minimal θ-chain whose three

minimal chains can be chosen to have common endpoints $N(u) \setminus \{v\}$,

(ii) $\delta(G, e) \leq -\frac{3}{2}$, *and*

(iii) $(\delta(G, e), \widehat{\delta}(G, e)) = (-\frac{3}{2}, \frac{3}{2})$ *if and only if* $G - e$ *is a minimal* θ*-chain and* e *joins two nonadjacent vertices of a 4-cycle in* $G - e$.

Proof. Let $N(u) \setminus \{v\} = \{x, y\}$, the set of endpoints of f_u. Let C_1, C_2 denote the two chains of (G_u, f_u) with common endpoints $\{x, y\}$, and let C_3 denote the subcubic chain $x(xu)u(uy)y$. Note that $n(G) = 2 + \sum_{i=1}^{3} n(\overline{C_i})$, $n_2(G) = -2 + \sum_{i=1}^{3} n_2(\overline{C_i})$ (since the $\overline{C_i}$'s do not account for the edge e), and $\overline{C_3}$ is a loop. Let i_1, i_2, i_3 be a permutation of $[3]$ such that $\delta(\overline{C_{i_1}}, e_{C_{i_1}}) \leq \delta(\overline{C_{i_2}}, e_{C_{i_2}}) \leq \delta(\overline{C_{i_3}}, e_{C_{i_3}})$.

Consider a triple (F_1, F_2, F_3) such that $F_{i_1} \in \mathcal{E}(\overline{C_{i_1}}, e_{C_{i_1}})$, $F_{i_2} \in \mathcal{E}(\overline{C_{i_2}}, e_{C_{i_2}})$, and $F_{i_3} \in \widehat{\mathcal{E}}(\overline{C_{i_3}}, e_{C_{i_3}})$. Let $F \in \widehat{\mathcal{E}}(G, e)$ be obtained from $F_1 \cup F_2 \cup F_3$ by merging the cycles in F_{i_1}, F_{i_2} through $e_{C_{i_1}}, e_{C_{i_2}}$. Then $\text{exc}(F) - 2 = (\text{exc}(F_{i_1}) - 2) + (\text{exc}(F_{i_2}) - 2) + \text{exc}(F_{i_3})$; so

$$\begin{aligned}\widehat{\text{exc}}(G, e) - 2 &= \text{exc}(\overline{C_{i_1}}, e_{C_{i_1}}) + \text{exc}(\overline{C_{i_2}}, e_{C_{i_2}}) + \widehat{\text{exc}}(\overline{C_{i_3}}, e_{C_{i_3}}) \\ &= \frac{n(G) + n_2(G)}{4} + \delta(\overline{C_{i_1}}, e_{C_{i_1}}) + \delta(\overline{C_{i_2}}, e_{C_{i_2}}) + \widehat{\delta}(\overline{C_{i_3}}, e_{C_{i_3}}).\end{aligned}$$

Since Theorem 3.1.4 holds for $(\overline{C_i}, e_{C_i})$ for each $i \in [3]$ (by assumption), we have

$$\widehat{\delta}(\overline{C_{i_3}}, e_{C_{i_3}}) \leq -\delta(\overline{C_{i_3}}, e_{C_{i_3}}) \leq -\delta(\overline{C_{i_2}}, e_{C_{i_2}})$$

and $\delta(\overline{C_i}, e_{C_i}) \leq -\frac{1}{2}$ for $i \in [3]$, which gives

$$\widehat{\text{exc}}(G, e) - 2 \leq \frac{n(G) + n_2(G)}{4} + \delta(\overline{C_{i_1}}, e_{C_{i_1}}) \leq \frac{n(G) + n_2(G)}{4} - \frac{1}{2}.$$

Therefore, $\widehat{\text{exc}}(G, e) \leq \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$, and $\widehat{\delta}(G, e) \leq \frac{3}{2}$.

Suppose $\widehat{\delta}(G, e) = \frac{3}{2}$. Then the above inequalities hold with equality. Hence, $-\frac{1}{2} =$

$\delta(\overline{C_{i_1}}, e_{C_{i_1}}) = \delta(\overline{C_{i_2}}, e_{C_{i_2}}) = \delta(\overline{C_{i_3}}, e_{C_{i_3}})$. Since Theorem 3.1.4 holds for all $(\overline{C_i}, e_{C_i})$ (by assumption), $(\overline{C_i}, e_{C_i})$ is tight (hence minimal) for all $i \in [3]$. Therefore, $G-e$ is a minimal θ-chain with its three chains having common endpoints $N(u) \setminus \{v\}$.

Now suppose $G - e$ is a minimal θ-chain with the three minimal chains C_1, C_2, C_3 with common endpoints $N(u) \setminus \{v\}$. Let $F \in \widehat{\mathcal{E}}(G, e)$. If F contains a cycle through two of C_1, C_2, C_3, then the above argument shows $\text{exc}(F) = \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$. So we just need to show that if F does not contain a cycle through any of C_1, C_2, C_3, then $\text{exc}(F) \geq \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$. Indeed, such F when restricted to $(\overline{C_i}, e_{C_i})$ for $i \in [3]$ gives a triple (F_1, F_2, F_3) such that $F_i \in \widehat{\mathcal{E}}(\overline{C_i}, e_{C_i})$ for each $i \in [3]$, and $\text{exc}(F) = 2 + \sum_{i=1}^{3} \text{exc}(F_i)$ (since the two vertices of $N(u) \setminus \{v\}$ are isolated in F). So

$$
\begin{aligned}
\text{exc}(F) &\geq 2 + \sum_{i=1}^{3} \widehat{\text{exc}}(\overline{C_i}, e_{C_i}) \\
&= 2 + \sum_{i=1}^{3} \left(\frac{n(\overline{C_i}) + n_2(\overline{C_i})}{4} + \widehat{\delta}(\overline{C_i}, e_{C_i}) \right) \\
&= \frac{n(G) + n_2(G)}{4} + 2 + \sum_{i=1}^{3} \widehat{\delta}(\overline{C_i}, e_{C_i}) \\
&= \frac{n(G) + n_2(G)}{4} + \frac{7}{2}.
\end{aligned}
$$

The last equality holds since $\widehat{\delta}(\overline{C_i}, e_{C_i}) = \frac{1}{2}$ for each $i \in [3]$, completing the proof of (i).

We now prove (ii) and (iii). Let us assume without loss of generality that $v \in V(C_1)$, and write $C_1 = xe_0B_1e_1B_2 \dots B_ke_ky$ with chain-blocks $(\overline{B_i}, \overline{e_i})$. Let $\ell \in [k]$ denote the unique index such that $v \in V(B_\ell)$. By symmetry, we may assume that $\sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \leq \sum_{j=\ell+1}^{k} \delta(\overline{B_j}, \overline{e_j})$. Then, by the assumption that Theorem 3.1.4 holds for each $(\overline{B_j}, \overline{e_j})$, we have

$$
\sum_{j=\ell+1}^{k} \widehat{\delta}(\overline{B_j}, \overline{e_j}) \leq \sum_{j=\ell+1}^{k} (-\delta(\overline{B_j}, \overline{e_j})) \leq - \left(\sum_{i=1}^{\ell-1} \delta(\overline{B_i}, \overline{e_i}) \right). \tag{3.5}
$$

Consider the tuple of even covers $(F_1, \dots, F_k, F^2)$, where $F_i \in \mathcal{E}(\overline{B_i}, \overline{e_i})$ for $i \in [\ell-1]$, $F_\ell \in \mathcal{E}(B_\ell + x'v, x'v)$ where x' is the endpoint of $e_{\ell-1}$ in B_ℓ, $F_j \in \widehat{\mathcal{E}}(\overline{B_j}, e_{B_j})$ for $j =$

$\ell+1,\dots,k$, and $F^2 \in \mathcal{E}(\overline{C_2}, e_{C_2})$. This corresponds to an even cover $F \in \mathcal{E}(G,e)$ containing a cycle through all of $xe_0B_1\dots B_{\ell-1}e_{\ell-1}$, e, uy, and C_2, such that

$$\mathrm{exc}(F)-2=\sum_{i=1}^{\ell-1}(\mathrm{exc}(F_i)-2)+(\mathrm{exc}(F_\ell)-2)+\sum_{j=\ell+1}^{k}\mathrm{exc}(F_j)+(\mathrm{exc}(F^2)-2).$$

Since

$$n(G)=\sum_{i=1}^{\ell-1}n(\overline{B_i})+n(B_\ell+x'v)+\sum_{j=\ell+1}^{k}n(\overline{B_j})+n(\overline{C_2})+3, \text{ and}$$
$$n_2(G)=\sum_{i=1}^{\ell-1}n_2(\overline{B_i})+n_2(B_\ell+x'v)+\sum_{j=\ell+1}^{k}n_2(\overline{B_j})+n_2(\overline{C_2})-1,$$

we have

$$\begin{aligned}
\mathrm{exc}(G,e) &\le \sum_{i=1}^{\ell-1}\mathrm{exc}(\overline{B_i},\overline{e_i})+\mathrm{exc}(B_\ell+x'v,x'v)+\sum_{j=\ell+1}^{k}\widehat{\mathrm{exc}}(\overline{B_j},\overline{e_j})+\mathrm{exc}(\overline{C_2},e_{C_2})\\
&=\frac{n(G)+n_2(G)}{4}-\frac{1}{2}+\left(\sum_{i=1}^{\ell-1}\delta(\overline{B_i},\overline{e_i})\right)+\delta(B_\ell+x'v,x'v)\\
&+\left(\sum_{j=\ell+1}^{k}\widehat{\delta}(\overline{B_j},\overline{e_j})\right)+\delta(\overline{C_2},e_{C_2})\\
&\le\frac{n(G)+n_2(G)}{4}-\frac{1}{2}+\delta(B_\ell+x'v,x'v)+\delta(\overline{C_2},e_{C_2}) \qquad \text{(by (3.5))}\\
&\le\frac{n(G)+n_2(G)}{4}-\frac{3}{2},
\end{aligned}$$

where the last inequality follows as by our assumption Theorem 3.1.4 holds for $(B_\ell+x'v,x'v)$ and $(\overline{C_2},e_{C_2})$. Hence $\delta(G,e)\le -\frac{3}{2}$ and (ii) holds.

To prove (iii), suppose $(\delta(G,e),\widehat{\delta}(G,e))=(-\frac{3}{2},\frac{3}{2})$. Then equality holds above, so we have $\delta(B_\ell+x'v,x'v)=\delta(\overline{C_2},e_{C_2})=-\frac{1}{2}$. Moreover, C_1 and C_2 are minimal chains (by (i)), which implies $k=\ell=1$ and $\delta(\overline{B_\ell},\overline{e_\ell})=\delta(\overline{C_1},e_{C_1})=-\frac{1}{2}$ (by Proposition 3.1.1). So $\delta(\overline{B_\ell},\overline{e_\ell})+\delta(B_\ell+x'v,x'v)=-1$. Now B_ℓ is a single vertex; otherwise, by applying Lemma 3.2.2 to B_ℓ, x', the other endpoint y' of $\overline{e_\ell}$, and v, we obtain $\delta(\overline{B_\ell},\overline{e_\ell})+\delta(B_\ell+$

$x'v, x'v) = \delta(B_\ell + x'y', x'y') + \delta(B_\ell + x'v, x'v) \leq -2$, a contradiction. Therefore, we have $B_\ell = \{x'\} = \{v\}$, and e joins two nonadjacent vertices of the 4-cycle $xvyux$. $\square$

We conclude this section with a lemma bounding $\widehat{\delta}(G, e)$, which proves statement **(T2)** of Theorem 3.1.4, assuming Theorem 3.1.4 for smaller graphs.

Lemma 3.2.4. *Let G be a 2-connected subcubic graph with $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected. Assume that Theorem 3.1.4 holds for graphs with fewer than $n(G)$ vertices. Then $\widehat{\delta}(G, e) \leq \frac{3}{2}$, with equality if and only if (G_u, f_u) is a minimal rooted θ-chain, where G_u is the graph obtained from $G - e$ by suppressing u into an edge f_u.*

Proof. Since $G - e$ is 2-connected, both u and v have degrees 3. Define G_u, f_u as stated in the lemma. We claim that

$$\widehat{\delta}(G, e) = \min\{\delta(G_u, f_u) + 2, \widehat{\delta}(G_u, f_u) + 1\}. \tag{3.6}$$

Indeed, there is a bijective correspondence between $\widehat{\mathcal{E}}(G, e)$ and $\mathcal{E}(G_u)$ obtained as follows. If $F \in \widehat{\mathcal{E}}(G, e)$ contains a cycle through u, then we obtain $F_u \in \mathcal{E}(G_u, f_u)$ by suppressing u in F, and we have $\text{exc}(F) = \text{exc}(F_u)$. Otherwise, if u is an isolated vertex in F, then we obtain $F_u \in \widehat{\mathcal{E}}(G_u, f_u)$ by removing u from F, and we have $\text{exc}(F) = \text{exc}(F_u) + 1$. Since $n(G) + n_2(G) = n(G_u) + n_2(G_u)$, (3.6) follows from the definitions of $\delta, \widehat{\delta}$.

It follows from (3.6) that $\widehat{\delta}(G, e) \leq \delta(G_u, f_u) + 2 \leq \frac{3}{2}$ by the assumption that Theorem 3.1.4 holds for (G_u, f_u). Moreover, $\widehat{\delta}(G, e) = \frac{3}{2}$ if and only if $\delta(G_u, f_u) = -\frac{1}{2}$ and $\widehat{\delta}(G_u, f_u) = \frac{1}{2}$, which is equivalent to (G_u, f_u) being a minimal rooted θ-chain by Lemma 3.2.1. $\square$

3.3 Proof of Theorem 3.1.4

We proceed by induction on $n(G)$. Note that **(T4)** is implied by **(T1)** and **(T2)**: If $\delta(G, e) \leq -1$ and $\widehat{\delta}(G, e) \leq 1$, then **(T4)** holds. Otherwise, we have $\delta(G, e) = -\frac{1}{2}$ or $\widehat{\delta}(G, e) = \frac{3}{2}$. In the former case, **(T4)** follows from **(T1)** and Lemma 3.2.1; in the latter case, **(T4)** follows from **(T2)** and Lemma 3.2.3. Also note that Lemmas 3.2.3 and 3.2.4 imply **(T2)**. Therefore, it suffices to prove **(T1)** and **(T3)**.

If $G - \{u, v\}$ is disconnected, then **(T1)** and **(T3)** both hold by Lemma 3.2.1. So we may assume that $G - \{u, v\}$ is connected. It now suffices to show that $\delta(G, e) \leq -1$ and that if equality holds, then one of the outcomes of **(T3)** holds.

Claim 3.3.1. *We may assume that G is simple.*

Proof. Since $G - e$ is simple, if G is not simple, then there is exactly one edge e^* parallel with e. Let G' be the graph obtained from G by suppressing $\{u, v\}$ to an edge e'.

Then $n(G) = n(G') + 2$ and $n_2(G) = n_2(G')$. By the inductive hypothesis, we have $\delta(G', e') \leq -\frac{1}{2}$. But every even cover $F' \in \mathcal{E}(G', e')$ gives an even cover $F \in \mathcal{E}(G, e)$ with the same excess, so

$$\begin{aligned}
\delta(G, e) &= \min_{F \in \mathcal{E}(G,e)} \operatorname{exc}(F) - 2 - \frac{n(G) + n_2(G)}{4} \\
&\leq \min_{F' \in \mathcal{E}(G',e')} \operatorname{exc}(F') - 2 - \frac{n(G') + n_2(G') + 2}{4} \\
&= \delta(G', e') - \frac{1}{2} \\
&\leq -1.
\end{aligned}$$

Now suppose $\delta(G, e) = -1$. Then both inequalities above are tight; in particular, we have $\delta(G', e') = -\frac{1}{2}$, and by the inductive hypothesis, G' is a loop or (G', e') is a balanced tight rooted θ-chain. If G' is a loop then (G, e) satisfies (b) of **(T3)**. So assume that (G', e') is a balanced tight rooted θ-chain, and let C_1, C_2 denote the two chains of (G', e').

Then a pair of even covers F_1, F_2 where $F_i \in \mathcal{E}(\overline{C_i}, e_{C_i})$ for each $i \in [2]$ gives an even cover $F \in \mathcal{E}(G, e)$ by combining the two cycles of F_i through e_{C_i} and adding the cycle with edge set $\{e, e^*\}$, with

$$\text{exc}(F) - 2 = (\text{exc}(F_1) - 2) + (\text{exc}(F_2) - 2) + 2.$$

Since $n(G) = n(\overline{C_1}) + n(\overline{C_2}) + 4$ and $n_2(G) = n_2(\overline{C_1}) + n_2(\overline{C_2})$, we have

$$\begin{aligned}\text{exc}(G, e) &\leq \text{exc}(\overline{C_1}, e_{C_1}) + \text{exc}(\overline{C_2}, e_{C_2}) + 2\\ &= \frac{n(G) + n_2(G)}{4} + 1 + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_1}, e_{C_1}),\end{aligned}$$

so $\delta(G, e) \leq 1 + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2})$. Thus, we have $\delta(\overline{C_i}, e_{C_i}) \in \{-\frac{1}{2}, -1\}$ for each $i \in [2]$; in other words, (G', e') is a near-minimal rooted θ-chain. So (G, e) satisfies (b) of **(T3)**. ∎

Claim 3.3.2. *We may assume that e is not in any 2-edge-cut of G.*

Proof. Suppose there is an edge e' such that $\{e, e'\}$ is a 2-edge-cut of G. Let C be a subcubic chain of G with end edges e, e'. By Proposition 3.1.2 and by the inductive hypothesis applied to $(G/C, e_{G/C})$ and $(\overline{C}, e_C)$, we have

$$\delta(G, e) = \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C) \leq -1.$$

Moreover, if $\delta(G, e) = -1$, then $\delta(G/C, e_{G/C}) = \delta(\overline{C}, e_C) = -\frac{1}{2}$, so $(G/C, e_{G/C})$ and $(\overline{C}, e_C)$ are loops or balanced tight rooted θ-chains and (c) of **(T3)** holds for (G, e). ∎

By Claim 3.3.2, let u_1, u_2 denote the two neighbors of u distinct from v, and let v_1, v_2 denote the two neighbors of v distinct from u. Moreover, there exist two disjoint paths P_1, P_2 from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ in $G - \{u, v\}$. We may assume without loss of generality that the set of endpoints of P_i is $\{u_i, v_i\}$, $i \in [2]$.

Let S denote the set of all cut edges in $G - \{u, v\}$. Then each component of $G - \{u, v\} - S$ is either an isolated vertex or 2-connected.

Claim 3.3.3. *For each $i \in [2]$, there is a unique component Z_i of $G - \{u, v\} - S$, such that there are three paths in $G - \{u, v\}$ from Z_i to $\{u_i, v_i, u_{3-i}\}$, pairwise disjoint except possibly at their endpoints in Z_i, and there are three paths in $G - \{u, v\}$ from Z_i to $\{u_i, v_i, v_{3-i}\}$, pairwise disjoint except possibly at their endpoints in Z_i. See Figures 3.4 and 3.5.*

Proof. By symmetry, it suffices to prove the claim for $i = 1$. First, we show that there is an unique component Z_1 of $G - \{u, v\} - S$ such that there are three paths in $G - \{u, v\}$ from Z_1 to $\{u_1, v_1, u_2\}$, pairwise disjoint except possibly at their endpoints in Z_1. Indeed, if there were two distinct such components Z, Z', they are by definition separated by a cut-edge $s \in S$ of $G - \{u, v\}$. But $G - \{u, v\} - s$ has exactly two connected components, one of which contains at least two of $\{u_1, v_1, u_2\}$, so one of Z, Z' is separated from two vertices of $\{u_1, v_1, u_2\}$ by a cut-edge, contradicting the assumptions on Z, Z'.

Similarly, there is a unique connected component Z'_1 of $G - \{u, v\} - S$ such that there are three paths in $G - \{u, v\}$ from Z'_1 to $\{u_1, v_1, v_2\}$, pairwise disjoint except possibly at their endpoints in Z'_1. We now show that $Z_1 = Z'_1$. Otherwise, there is a cut edge s of $G - \{u, v\}$ separating Z_1 from Z'_1. Then the two connected components of $G - \{u, v\} - s$ each contain exactly one of $\{u_1, v_1\}$ and exactly one of $\{u_2, v_2\}$. But this implies that $\{e, s\}$ is a 2-edge-cut in G, contradicting Claim 3.3.2. ∎

There are two cases to consider: either $Z_1 \neq Z_2$ or $Z_1 = Z_2$. For $i \in [2]$, let u'_i (respectively, v'_i) denote the vertex of Z_i that is the endpoint of a (possibly trivial) path in $G - \{u, v\}$ from u_i (respectively, v_i) to Z_i that is internally disjoint from $Z_1 \cup Z_2$. Note that u'_i and v'_i are uniquely determined. For $i \in [2]$, let U_i denote the unique (possibly trivial) subcubic chain of $G - \{v, u_{3-i}\}$ with endpoints $\{u, u'_i\}$, and let V_i denote the unique subcubic chain of $G - \{u, v_{3-i}\}$ with endpoints $\{v, v'_i\}$.

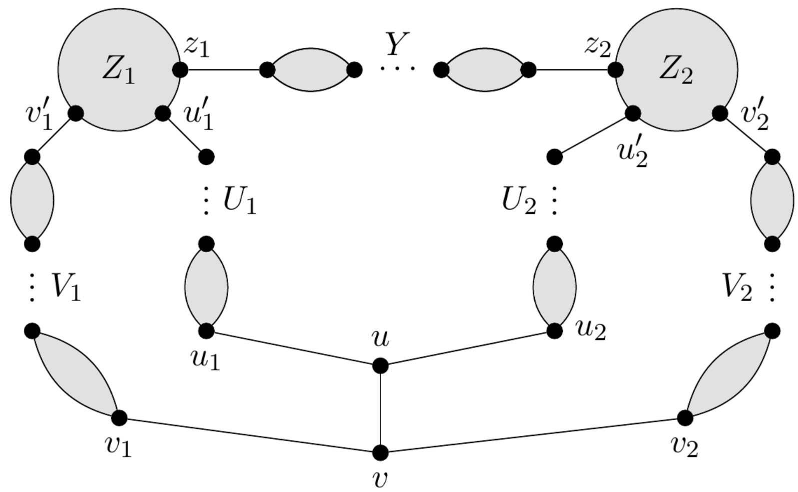

Figure 3.4: $Z_1 \neq Z_2$

Case 1: $Z_1 \neq Z_2$.

There is a cut-edge separating Z_1 and Z_2 in $G - \{u, v\}$ and there is a unique subcubic chain Y of $G - \{u, v\}$ with an endpoint $z_i \in Z_i$ for each $i \in [2]$, internally disjoint from $Z_1 \cup Z_2$. Then G is the union of $U_1, U_2, V_1, V_2, Z_1, Z_2, Y$, and the edge $e = uv$. We have, for $i, j \in [2]$,

$$n(G) = n(\overline{U_1}) + n(\overline{U_2}) + n(\overline{V_1}) + n(\overline{V_2}) + n(Z_1 + u_i' z_1) + n(Z_2 + v_j' z_2) + n(\overline{Y}) + 2,$$

$$n_2(G) = n_2(\overline{U_1}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(\overline{V_2}) + n_2(Z_1 + u_i' z_1) + n_2(Z_2 + v_j' z_2) + n_2(\overline{Y}) - 2.$$

Suppose $F \in \mathcal{E}(G, e)$ goes through U_1, Y, and V_2. Then there is a correspondence between F and the tuple $(F_{U_1}, F_{Z_1}, F_Y, F_{Z_2}, F_{V_2}, F_{U_2}, F_{V_1})$, where

- $F_{U_1} \in \mathcal{E}(\overline{U_1}, e_{U_1}),\ F_{Z_1} \in \mathcal{E}(Z_1 + u_1' z_1, u_1' z_1),\ F_Y \in \mathcal{E}(\overline{Y}, e_Y),\ F_{Z_2} \in \mathcal{E}(Z_2 + v_2' z_2, v_2' z_2),\ F_{V_2} \in \mathcal{E}(\overline{V_2}, e_{V_2})$, and
- $F_{U_2} \in \widehat{\mathcal{E}}(\overline{U_2}, e_{U_2}),\ F_{V_1} \in \widehat{\mathcal{E}}(\overline{V_1}, e_{V_1})$.

This gives

$$
\begin{aligned}
\operatorname{exc}(G,e) &\le \operatorname{exc}(\overline{U_1}, e_{U_1}) + \operatorname{exc}(Z_1 + u_1'z_1, u_1'z_1) + \operatorname{exc}(\overline{Y}, e_Y) + \operatorname{exc}(Z_2 + v_2'z_2, v_2'z_2) \\
&\quad + \operatorname{exc}(\overline{V_2}, e_{V_2}) + \widehat{\operatorname{exc}}(\overline{U_2}, e_{U_2}) + \widehat{\operatorname{exc}}(\overline{V_1}, e_{V_1}) \\
&= \frac{n(G) + n_2(G)}{4} + \delta(\overline{U_1}, e_{U_1}) + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) \\
&\quad + \delta(Z_2 + v_2'z_2, v_2'z_2) + \delta(\overline{V_2}, e_{V_2}) + \widehat{\delta}(\overline{U_2}, e_{U_2}) + \widehat{\delta}(\overline{V_1}, e_{V_1}),
\end{aligned}
$$

hence

$$
\begin{aligned}
\delta(G,e) &\le \delta(\overline{U_1}, e_{U_1}) + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) + \delta(\overline{V_2}, e_{V_2}) \\
&\quad + \widehat{\delta}(\overline{U_2}, e_{U_2}) + \widehat{\delta}(\overline{V_1}, e_{V_1}).
\end{aligned} \tag{3.7}
$$

Similarly, by considering an even cover in $\mathcal{E}(G,e)$ through U_2, Y, and V_1, we obtain

$$
\begin{aligned}
\delta(G,e) &\le \delta(\overline{U_2}, e_{U_2}) + \delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1) + \delta(\overline{V_1}, e_{V_1}) \\
&\quad + \widehat{\delta}(\overline{U_1}, e_{U_1}) + \widehat{\delta}(\overline{V_2}, e_{V_2}).
\end{aligned} \tag{3.8}
$$

Now suppose $\delta(G,e) \geq -1$. Then

$$
\begin{aligned}
-1 &\leq \delta(\overline{U_1}, e_{U_1}) + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) + \delta(\overline{V_2}, e_{V_2}) \\
&\quad + \widehat{\delta}(\overline{U_2}, e_{U_2}) + \widehat{\delta}(\overline{V_1}, e_{V_1}) \\
&\hspace{25em} \text{(by (3.7))} \\
&\leq -\big(\widehat{\delta}(\overline{U_1}, e_{U_1}) + \widehat{\delta}(\overline{V_2}, e_{V_2}) + \delta(\overline{U_2}, e_{U_2}) + \delta(\overline{V_1}, e_{V_1})\big) \\
&\hspace{20em} \text{(by inductive hypothesis)} \\
&\quad + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) \\
&= -\big(\widehat{\delta}(\overline{U_1}, e_{U_1}) + \widehat{\delta}(\overline{V_2}, e_{V_2}) + \delta(\overline{U_2}, e_{U_2}) + \delta(\overline{V_1}, e_{V_1})\big) \\
&\quad - \big(\delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1)\big) \\
&\quad + \big(\delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1)\big) \\
&\quad + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) \\
&\leq 1 + \delta(Z_1 + u_1'z_1, u_1'z_1) + \delta(\overline{Y}, e_Y) + \delta(Z_2 + v_2'z_2, v_2'z_2) \qquad \text{(by (3.8))} \\
&\quad + \delta(Z_2 + u_2'z_2, u_2'z_2) + \delta(\overline{Y}, e_Y) + \delta(Z_1 + v_1'z_1, v_1'z_1).
\end{aligned}
$$

This gives

$$
\begin{aligned}
-2 \leq \delta(Z_1 + u_1'z_1, u_1'z_1) &+ \delta(Z_2 + v_2'z_2, v_2'z_2) + \delta(Z_2 + u_2'z_2, u_2'z_2) \\
&+ \delta(Z_1 + v_1'z_1, v_1'z_1) + 2\delta(\overline{Y}, e_Y) \leq -2,
\end{aligned}
$$

since by the inductive hypothesis, the all terms are each at most $-\frac{1}{2}$ except $\delta(\overline{Y}, e_Y) = 0$ when Y is a trivial chain. Hence, $\delta(G,e) = -1$,

$$
\delta(Z_1 + u_1'z_1, u_1'z_1) = \delta(Z_2 + v_2'z_2, v_2'z_2) = \delta(Z_1 + u_2'z_2, u_2'z_2) = \delta(Z_1 + v_1'z_1, v_1'z_1) = -\frac{1}{2},
$$

and $\delta(\overline{Y}, e_Y) = 0$ (i.e., Y is a trivial chain). By Lemma 3.2.2, Z_1 and Z_2 are single vertices. So for $i, j \in [2]$, $\delta(Z_i + u_j'z_i, u_j'z_i) = \delta(Z_i + v_j'z_i, v_j'z_i) = -\frac{1}{2}$. Hence, from (3.7) and (3.8), and by the inductive hypothesis, we have

$$\delta(\overline{U_i}, e_{U_i}) = \delta(\overline{V_j}, e_{V_j}) = 0$$

for each $i, j \in [2]$, so U_i, V_j are all trivial chains as well. This proves that $G \cong K_4$, satisfying (a) of **(T3)**.

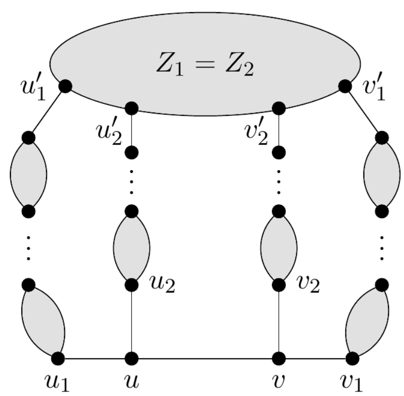

Figure 3.5: $Z_1 = Z_2$

Case 2: $Z_1 = Z_2$.

Let $Z := Z_1 = Z_2$. Then u_1', u_2', v_1', v_2' are distinct vertices (since G is subcubic and Z is 2-connected), and G is the union of U_1, U_2, V_1, V_2, Z, and the edge e. Note that

$$n(G) = n(\overline{U_1}) + n(\overline{U_2}) + n(\overline{V_1}) + n(\overline{V_2}) + n(Z + u_i'v_j') + 2$$
$$n_2(G) = n_2(\overline{U_1}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(\overline{V_2}) + n_2(Z + u_i'v_j') - 2.$$

For $i, j \in [2]$, let $F \in \mathcal{E}(G, e)$ be an even cover through U_i and V_j. This corresponds to a tuple $(F_{U_1}, F_{U_2}, F_{V_1}, F_{V_2}, F_Z)$ where

- $F_{U_i} \in \mathcal{E}(\overline{U_i}, e_{U_i}),\ F_{V_j} \in \mathcal{E}(\overline{V_j}, e_{V_j}),\ F_Z \in \mathcal{E}(Z + u_i'v_j', u_i'v_j')$, and
- $F_{U_{3-i}} \in \widehat{\mathcal{E}}(\overline{U_{3-i}}, e_{U_{3-i}}),\ F_{V_{3-j}} \in \widehat{\mathcal{E}}(\overline{V_{3-j}}, e_{V_{3-j}}),$

which gives

$$\begin{aligned}\mathrm{exc}(G,e) &\leq \mathrm{exc}(\overline{U_i}, e_{U_i}) + \mathrm{exc}(\overline{V_j}, e_{V_j}) + \mathrm{exc}(Z + u_i'v_j', u_i'v_j') \\ &\quad + \widehat{\mathrm{exc}}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\mathrm{exc}}(\overline{V_{3-j}}, e_{V_{3-j}}) \\ &= \frac{n(G)+n_2(G)}{4} + \delta(\overline{U_i}, e_{U_i}) + \delta(\overline{V_j}, e_{V_j}) + \delta(Z + u_i'v_j', u_i'v_j') \\ &\quad + \widehat{\delta}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V_{3-j}}, e_{V_{3-j}}).\end{aligned}$$

Hence, for all $i, j \in [2]$,

$$\delta(G,e) \leq \delta(\overline{U_i}, e_{U_i}) + \delta(\overline{V_j}, e_{V_j}) + \delta(Z + u_i'v_j', u_i'v_j') + \widehat{\delta}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V_{3-j}}, e_{V_{3-j}}) \tag{3.9}$$

We now show that $\delta(G,e) \leq -\frac{3}{2}$, which completes the proof of Theorem 3.1.4. Suppose to the contrary that $\delta(G,e) \geq -1$. Then by (3.9) and the inductive hypothesis,

$$\begin{aligned}-1 &\leq \delta(\overline{U_i}, e_{U_i}) + \delta(\overline{V_j}, e_{V_j}) + \widehat{\delta}(\overline{U_{3-i}}, e_{U_{3-i}}) + \widehat{\delta}(\overline{V_{3-j}}, e_{V_{3-j}}) + \delta(Z + u_i'v_j', u_i'v_j') && \text{(by (3.9))} \\ &\leq -\big(\widehat{\delta}(\overline{U_i}, e_{U_i}) + \widehat{\delta}(\overline{V_j}, e_{V_j}) + \delta(\overline{U_{3-i}}, e_{U_{3-i}}) + \delta(\overline{V_{3-j}}, e_{V_{3-j}})\big) && \text{(by } \mathbf{(T4)}\text{)} \\ &\quad + \delta(Z + u_i'v_j', u_i'v_j') \\ &= -\big(\widehat{\delta}(\overline{U_i}, e_{U_i}) + \widehat{\delta}(\overline{V_j}, e_{V_j}) + \delta(Z + u_{3-i}'v_{3-j}', u_{3-i}'v_{3-j}') + \delta(\overline{U_{3-i}}, e_{U_{3-i}}) \\ &\quad + \delta(\overline{V_{3-j}}, e_{V_{3-j}})\big) + \delta(Z + u_i'v_j', u_i'v_j') + \delta(Z + u_{3-i}'v_{3-j}', u_{3-i}'v_{3-j}') \\ &\leq 1 + \delta(Z + u_i'v_j', u_i'v_j') + \delta(Z + u_{3-i}'v_{3-j}', u_{3-i}'v_{3-j}'). && \text{(by (3.9))}\end{aligned}$$

Hence for $i, j \in [2]$,

$$-2 \leq \delta(Z + u_i'v_j', u_i'v_j') + \delta(Z + u_{3-i}'v_{3-i}', u_{3-j}'v_{3-j}') \tag{3.10}$$

On the other hand, applying Lemma 3.2.2 to u'_i, v'_1, v'_2 and v'_j, u'_1, u'_2, we have for all $i, j \in [2]$

$$\begin{aligned}
&\delta(Z + u'_i v'_1, u'_i v'_1) + \delta(Z + u'_i v'_2, u'_i v'_2) \leq -2 \text{ and} \\
&\delta(Z + u'_1 v'_j, u'_1 v'_j) + \delta(Z + u'_2 v'_j, u'_2 v'_j) \leq -2.
\end{aligned} \tag{3.11}$$

Now, setting $i = j = 1$ and setting $i = 1$ and $j = 2$ in (3.10), we have

$$-4 \leq \delta(Z + u'_1 v'_1, u'_1 v'_1) + \delta(Z + u'_2 v'_2, u'_2 v'_2) + \delta(Z + u'_1 v'_2, u'_1 v'_2) + \delta(Z + u'_2 v'_1, u'_2 v'_1).$$

On the other hand, setting $i = 1$ and $i = 2$ in the first inequality of (3.11), we have

$$\delta(Z + u'_1 v'_1, u'_1 v'_1) + \delta(Z + u'_1 v'_2, u'_1 v'_2) + \delta(Z + u'_2 v'_1, u'_2 v'_1) + \delta(Z + u'_2 v'_2, u'_2 v'_2) \leq -4.$$

We thus have equality everywhere. In particular, $\delta(G, e) = -1$ and we have equality in (3.10) and (3.11), which implies that for all $i, j \in [2]$,

$$\delta(Z + u'_i v'_j, u'_i v'_j) = -1. \tag{3.12}$$

Since $Z + u'_i v'_j$ has at least two vertices of degree 2 (namely u'_{3-i} and v'_{3-j}), it is not isomorphic to K_4. Moreover, since Z is 2-connected, $u'_i v'_j$ is not contained in any 2-edge-cut in $Z + u'_i v'_j$. So each $(Z + u'_i v'_j, u'_i v'_j)$ satisfies (b) or (d) of **(T3)**.

We claim that $u'_i v'_j \notin E(Z)$ for all $i, j \in [2]$ (hence $(Z + u'_i v'_j, u'_i v'_j)$ satisfies (d) of **(T3)**). For, suppose without loss of generality that $u'_1 v'_1 \in E(Z)$.

By the inductive hypothesis, (b) of **(T3)** holds for $(Z + u'_1 v'_1, u'_1 v'_1)$, so suppressing $\{u'_1, v'_1\}$ in Z to an edge e' results in a graph Z' such that (Z', e') is a near-minimal rooted θ-chain. Let C_1, C_2 denote the two chains of (Z', e'). Assume without loss of generality that $v'_2 \in V(C_1)$. Since v'_2 has degree 2 in Z, it is in the interior of C_1, and this implies that $Z - \{u'_1, v'_2\}$ is connected and $v'_2 u'_1 \notin E(Z)$. Then $(Z + u'_1 v'_2, u'_1 v'_2)$ satisfies (d) of **(T3)**, which implies that $Z - \{u_1', v_2'\}$ is disconnected, a contradiction.

It follows that $(Z + u_i'v_j', u_i'v_j')$ satisfy (d) of **(T3)** for all $i, j \in [2]$, so $(Z + u_i'v_j', u_i'v_j')$ is a rooted θ-chain for all $i, j \in [2]$. Consider the rooted θ-chain $(Z + u_1'v_1', u_1'v_1')$. Since $(Z + u_1'v_2', u_1'v_2')$ (respectively, $(Z + u_2'v_1', u_2'v_1')$) is a rooted θ-chain, $\{v_2'\}$ (respectively, $\{u_2'\}$) is a block in one of the chains of $(Z + u_1'v_1', u_1'v_1')$. Let C_1 denote the subcubic chain of Z with end points $\{u_1', v_1'\}$ not containing v_2', and let C_2 denote the subcubic chain of Z with end points $\{u_1', v_2'\}$ not containing v_1'. Let D denote the subcubic chain of Z with end points $\{v_1', v_2'\}$ not containing u_1'.

Then for $j \in [2]$, $n(Z + u_1'v_j') = n(\overline{C_1}) + n(\overline{C_2}) + n(\overline{D}) + 3$ and $n_2(Z + u_1'v_j') = n_2(\overline{C_1}) + n_2(\overline{C_2}) + n_2(\overline{D}) + 1$. Thus for each $j \in [2]$, by forming an even cover in $\mathcal{E}(Z + u_1'v_j', u_1'v_j')$ using even covers from $\widehat{\mathcal{E}}(\overline{C_j}, e_{C_j})$, $\mathcal{E}(\overline{D}, e_D)$, and $\mathcal{E}(\overline{C_{3-j}}, e_{C_{3-j}})$, we obtain

$$\delta(Z + u_1'v_j', u_1'v_j') \leq -1 + \widehat{\delta}(\overline{C_j}, e_{C_j}) + \delta(\overline{D}, e_D) + \delta(\overline{C_{3-j}}, e_{C_{3-j}}).$$

Adding these two inequalities and using (3.12), we have

$$\begin{aligned} 0 &\leq \delta(\overline{C_1}, e_{C_1}) + \widehat{\delta}(\overline{C_1}, e_{C_1}) + 2\delta(\overline{D}, e_D) + \delta(\overline{C_2}, e_{C_2}) + \widehat{\delta}(\overline{C_2}, e_{C_2}) \\ &\leq 2\delta(\overline{D}, e_D) \end{aligned}$$

by **(T4)** applied to $(\overline{C_i}, e_{C_i})$. It follows that D is a trivial chain, and $v_1'v_2' \in E(Z)$.

By symmetry, $u_1'u_2' \in E(Z)$. Thus, $\{u_1'u_2', v_1v_2'\}$ is a 2-edge-cut in Z. Let D_1, D_2 denote the connected components of $Z - \{u_1'u_2', v_1'v_2'\}$ and (by relabeling u_1', u_2' if necessary) assume $u_i', v_i' \in V(D_i)$ for $i \in [2]$.

Then for $i, j \in [2]$, $n(Z + u_i'v_j', u_i'v_j') = n(\overline{D_1}, e_{D_1}) + n(\overline{D_2}, e_{D_2}) + 4$ and $n_2(Z + u_i'v_j', u_i'v_j') = n_2(\overline{D_1}, e_{D_1}) + n_2(\overline{D_2}, e_{D_2}) + 2$. Thus, by forming an even cover in $\mathcal{E}(Z +$

$u_i'v_j', u_i'v_j')$ using even covers from $\mathcal{E}(\overline{D_k}, e_{D_k})$ and $\widehat{\mathcal{E}}(\overline{D_{3-k}}, e_{D_{3-k}})$ for $k \in [2]$, we get

$$\delta(Z + u_i'v_j', u_i'v_j') \leq -\frac{3}{2} + \delta(\overline{D_k}, e_{D_k}) + \widehat{\delta}(\overline{D_{3-k}}, e_{D_{3-k}})$$

Adding these two inequalities and using (3.12) and **(T4)**, we have

$$1 \leq \delta(\overline{D_1}, e_{D_1}) + \widehat{\delta}(\overline{D_1}, e_{D_1}) + \delta(\overline{D_2}, e_{D_2}) + \widehat{\delta}(\overline{D_2}, e_{D_2}) \leq 0,$$

a contradiction. This completes the proof of Theorem 3.1.4.

3.4 Extremal Examples

In this section, we give a structural characterization of the extremal examples of Theorem 1.1.2. Recall that for a subcubic graph G and any edge $e \in E(G)$, we have

$$\begin{aligned} \mathrm{exc}(G) &= \min\{\mathrm{exc}(G, e) + 2,\ \widehat{\mathrm{exc}}(G, e)\} \\ &= \frac{n(G) + n_2(G)}{4} + \min\{\delta(G, e) + 2,\ \widehat{\delta}(G, e)\}. \end{aligned}$$

So if either $\delta(G, e) \leq -\frac{3}{2}$ or $\widehat{\delta}(G, e) \leq \frac{1}{2}$ for any edge $e \in E(G)$, then $\mathrm{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + \frac{1}{2}$. It follows that $\mathrm{exc}(G) = \frac{n(G)+n_2(G)}{4} + 1$ (equivalently, $\mathrm{tsp}(G) = \frac{5n(G)+n_2(G)}{4} - 1$) if and only if $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$ for all $e \in E(G)$.

Proposition 3.4.1. *Let G be a simple 2-connected subcubic graph and let e be an edge of G. Then $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$ if and only if either $G \cong K_4$ or G is a minimal θ-chain.*

Proof. Suppose $(\delta(G, e), \widehat{\delta}(G, e)) = (-1, 1)$. Since $\delta(G, e) = -1$, one of the four outcomes of **(T3)** holds. If $G \cong K_4$ then we are done. Since G is simple, (b) of **(T3)** cannot occur. Moreover, (d) of **(T3)** does not hold; otherwise, (G, e) is a simple rooted θ-chain and, by Lemma 3.2.1 (ii), $\widehat{\delta}(G, e) \leq \frac{3}{2} + \delta(\overline{C_1}, e_{C_1}) + \delta(\overline{C_2}, e_{C_2}) \leq 1/2$, a contradiction.

Thus (c) of **(T3)** holds: there exists $e' \in E(G)$ such that $\{e, e'\}$ is a 2-edge-cut in G and suppressing either subcubic chain C of G with end edges e, e' yields a loop or a balanced tight rooted θ-chain $(G/C, e_{G/C})$. Let C be a subcubic chain of G with end edges e, e'. Then by Proposition 3.1.2 and **(T4)**,

$$\begin{aligned}-1 = \delta(G, e) &= \delta(G/C, e_{G/C}) + \delta(\overline{C}, e_C) \\ &\leq -\big(\widehat{\delta}(G/C, e_{G/C}) + \widehat{\delta}(\overline{C}, e_C)\big) = -\widehat{\delta}(G, e) = -1.\end{aligned}$$

This implies that $(\delta(G/C, e_{G/C}), \widehat{\delta}(G/C, e_{G/C})) = (\delta(\overline{C}, e_C), \widehat{\delta}(\overline{C}, e_C)) = (-\frac{1}{2}, \frac{1}{2})$, and thus $(\overline{C}, e_C)$ and $(G/C, e_{G/C})$ are minimal rooted θ-chains (by Lemma 3.2.1 (iii)). Therefore, by definition, G is a minimal θ-chain (since it is the internally disjoint union of C and the two chains of $(G/C, e_{G/C})$, all of which are minimal). $\square$

To give an alternate structural characterization of minimal (rooted) θ-chains, we now describe an operation introduced in [19]. Let H be a graph and $v \in V(H)$ be a vertex of degree 2. A *$\diamond$-operation on H at v* deletes v from H, adds a 4-cycle D disjoint from $H - v$, and adds a matching between the neighbors of v and two nonadjacent vertices in D. See Figure 3.6. We say that a graph is *H-constructible* if it can be obtained from H by repeated $\diamond$-operations.

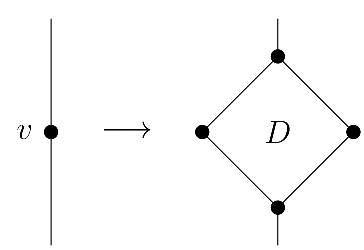

Figure 3.6: The $\diamond$-operation

It is observed in [19] that after each $\diamond$-operation, the excess of the new graph increases by 1 and the new quantity $\frac{n(G)+n_2(G)}{4}$ also increases by 1. We will consider $K_{2,3}$-constructible graphs and K_4^--constructible graphs, where K_4^- is the graph obtained from the complete graph K_4 by removing an edge. Note that $\mathrm{exc}(K_{2,3}) = \frac{n(K_{2,3})+n_2(K_{2,3})}{4} + 1$; thus, if G is $K_{2,3}$-constructible then $\mathrm{exc}(G) = \frac{n(G)+n_2(G)}{4} + 1$.

Proposition 3.4.2 (Dvořák et al. [19]). *Let G be a simple 2-connected subcubic graph. If $G \cong K_4$ or G is $K_{2,3}$-constructible, then $(\delta(G,e), \widehat{\delta}(G,e)) = (-1, 1)$.*

We show that the converse of Proposition 3.4.2 is also true, thereby giving a structural characterization of the extremal graphs for Theorem 1.1.2. First, we have an observation similar to Proposition 3.4.2. The *center* of K_4^- is the edge whose endpoints both have degree 3.

Proposition 3.4.3. *Let (G, e) be a simple minimal rooted θ-chain. Then G is K_4^--constructible, with the edge e corresponding to the center of K_4^-.*

Proof. By **(T1)** and Lemma 3.2.1 (iii), $(\delta(G,e), \widehat{\delta}(G,e)) = (-\frac{1}{2}, \frac{1}{2})$. Let C_1 and C_2 be the chains of (G, e). By the definition of a minimal rooted θ-chain, for each $i \in [2]$, we have $(\delta(\overline{C_i}, e_{C_i}), \widehat{\delta}(\overline{C_i}, e_{C_i})) = (-\frac{1}{2}, \frac{1}{2})$, so $(\overline{C_i}, e_{C_i})$ is either a loop or a minimal rooted θ-chain by (**(T1)**) and Lemma 3.2.1. If $(\overline{C_i}, e_{C_i})$ is not a loop, then by induction, it is K_4^--constructible with e_{C_i} corresponding to the center of K_4^-. It follows that (G, e) is K_4^--constructible with e corresponding to the center of K_4^-. □

Proposition 3.4.4. *Let G be a simple minimal θ-chain. Then G is $K_{2,3}$-constructible.*

Proof. By definition, there exists a choice of three chains C_1, C_2, C_3 of G with common endpoints such that G is the internally disjoint union $C_1 \cup C_2 \cup C_3$, and we have $(\delta(\overline{C_i}, e_{C_i}), \widehat{\delta}(\overline{C_i}, e_{C_i})) = (-\frac{1}{2}, \frac{1}{2})$ for each $i \in [3]$. If $G \cong K_{2,3}$, then we are done. So we may assume without loss of generality that $(\overline{C_1}, e_{C_1})$ is not a loop. Then it is a minimal rooted θ-chain by Lemma 3.2.1, and by Proposition 3.4.3, it is K_4^--constructible with the edge e_{C_1} corresponding to the center of K_4^-. On the other hand, $(G/C_1, e_{G/C_1})$ is by definition a minimal rooted θ-chain, so it is also K_4^--constructible by Proposition 3.4.3, with $e_{G/C}$ corresponding to the center of K_4^-. It follows that G is $K_{2,3}$-constructible. □

We thus have the following characterization of the extremal examples of Theorem 1.1.2.

Theorem 3.4.5. *Let G be a simple 2-connected subcubic graph. Then* $\mathrm{exc}(G) \leq \frac{n(G)+n_2(G)}{4}+1$*, with equality if and only if either $G \cong K_4$ or G is $K_{2,3}$-constructible.*

Proof. Let $e \in E(G)$. If $\delta(G,e) \leq -\frac{3}{2}$ or $\widehat{\delta}(G,e) \leq \frac{1}{2}$, then $\mathrm{exc}(G) \leq \frac{n(G)+n_2(G)}{4} + \frac{1}{2}$. Otherwise, we have $(\delta(G,e), \widehat{\delta}(G,e)) = (-1,1)$, or equivalently, $\mathrm{exc}(G) = \frac{n(G)+n_2(G)}{4}+1$. Now if $G \cong K_4$ or G is $K_{2,3}$-constructible, then $(\delta(G,e), \widehat{\delta}(G,e)) = (-1,1)$ by Proposition 3.4.2. Conversely, if $(\delta(G,e), \widehat{\delta}(G,e)) = (-1,1)$, then by Propositions 3.4.1 and 3.4.4, either $G \cong K_4$ or G is $K_{2,3}$-constructible. $\square$

3.5 The Algorithm

We now provide an algorithm for finding a TSP walk of length at most $\frac{5n(G)+n_2(G)}{4} - 1$ in any simple 2-connected subcubic graph G. This is achieved by following the proof of Theorem 3.1.4 to construct an even cover F of G with $\mathrm{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$. As noted by Dvořák et al. [19], modifying this even cover to our desired TSP walk takes linear time.

In the proof of Theorem 3.1.4, we often have a choice of routing a cycle through certain subcubic chains and not through others. For each such chain C, we "save" $\delta(\overline{C}, e_C)$ by going through C and incur a "cost" $\widehat{\delta}(\overline{C}, e_C)$ by not going through C. The key idea of Theorem 3.1.4 is that these costs and savings are (at worst) balanced, i.e. $\delta(\overline{C}, e_C) + \widehat{\delta}(\overline{C}, e_C) \leq 0$. Of course, for a given subcubic graph G and an edge e, we cannot efficiently compute $\delta(G,e)$ and $\widehat{\delta}(G,e)$ exactly (unless P=NP). Instead, we compute "worst-case" estimates

$$(\Delta(G,e), \widehat{\Delta}(G,e)) \in \left\{(-\tfrac{1}{2}, \tfrac{1}{2}),\ (-1,1),\ (-\tfrac{3}{2}, \tfrac{3}{2})\right\}$$

such that $(\delta(G,e), \widehat{\delta}(G,e)) \leq (\Delta(G,e), \widehat{\Delta}(G,e))$ (coordinate-wise).

The natural approach would be to determine exactly when $\delta(G,e) = -\frac{1}{2}$ or $\widehat{\delta}(G,e) = \frac{3}{2}$ using our characterization of the extremal examples in Theorem 3.1.4, and then assigning $(\Delta(G,e), \widehat{\Delta}(G,e))$ to be $(-\frac{1}{2}, \frac{1}{2})$ or $(-\frac{3}{2}, \frac{3}{2})$ respectively (and assign $(-1,1)$ in all other cases). To check whether (G, e) is a minimal rooted θ-chain (for example), we would need to first check that it is a rooted θ-chain (which takes linear time) and then recursively check that each of its two chains are also minimal, taking quadratic time overall. This approach would result in a cubic algorithm to produce the desired even covers.

It turns out that a much simpler linear-time estimate is sufficient, and yields a quadratic-time algorithm to find the desired even covers. Indeed, by Lemma 3.2.1, if (G, e) is a rooted θ-chain (regardless of whether it is tight or balanced), then we have $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{1}{2}, \frac{1}{2})$. And by Lemma 3.2.3, if $G - e$ is simple and 2-connected and (G_u, f_u) is a rooted θ-chain (where G_u is obtained from $G - e$ by suppressing an endpoint u to an edge f_u), then we have $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{3}{2}, \frac{3}{2})$.

We thus define an algorithm $\texttt{Scan}(G, e)$ to estimate $(\delta(G, e), \widehat{\delta}(G, e))$ as follows. If G is a loop or $G - e$ is 2-connected, $\texttt{Scan}(G, e)$ will assign

$$(\Delta(G, e), \widehat{\Delta}(G, e)) = \begin{cases} (-\frac{1}{2}, \frac{1}{2}) & \text{if } (G, e) \text{ is a loop or a rooted } \theta\text{-chain,} \\ (-\frac{3}{2}, \frac{3}{2}) & \text{if } (G_u, f_u) \text{ is a rooted } \theta\text{-chain,} \\ (-1, 1) & \text{otherwise.} \end{cases}$$

If $G - e$ is not 2-connected (and it is not a loop), then (G, e) can be written as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$ such that $k \geq 2$ (if $k = 1$, then $G - e = \overline{C} - e_C$ is 2-connected or an isolated vertex). In this case, our estimate on (G, e) will be the sum of the estimates of the chain-blocks $(\overline{B_i}, e_{B_i})$ of C:

$$(\Delta(G, e), \widehat{\Delta}(G, e)) = \sum_{i=1}^{k} (\Delta(\overline{B_i}, e_{B_i}), \widehat{\Delta}(\overline{B_i}, e_{B_i})).$$

For the remainder of this section, given a 2-connected subcubic graph G and an edge $e = uv \in E(G)$ such that $G - e$ is simple and has no cut-vertex, we let u_1, u_2 denote the two neighbors of u not equal to v, and denote by G_u the graph obtained by deleting e and suppressing u to an edge $f_u = u_1u_2$. Note that computing G_u and f_u takes constant time.

To resolve ambiguities in the choice of the vertex u in the edge $e = uv$ (in the case where $\widehat{\Delta}(G, e) = \frac{3}{2}$), we fix a linear ordering $\leq$ of the vertices throughout, and assume that $u \leq v$.

Proposition 3.5.1. *Let G be a subcubic graph and let $e = uv \in E(G)$ such that $G - e$ is simple. Then $\delta(G, e) \leq \Delta(G, e)$ and $\widehat{\delta}(G, e) \leq \widehat{\Delta}(G, e)$.*

Proof. First suppose G is a loop or $G - e$ is 2-connected. If (G, e) is a loop or a rooted θ-chain, then by Lemma 3.2.1, $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{1}{2}, \frac{1}{2}) = (\Delta(G, e), \widehat{\Delta}(G, e))$. If (G_u, f_u) is a rooted θ-chain, then by Lemma 3.2.3,

$$(\delta(G, e), \widehat{\delta}(G, e)) \leq (-\frac{3}{2}, \frac{3}{2}) = (\Delta(G, e), \widehat{\Delta}(G, e)).$$

Otherwise, by Theorem 3.1.4, we have $(\delta(G, e), \widehat{\delta}(G, e)) \leq (-1, 1) = (\Delta(G, e), \widehat{\Delta}(G, e))$.

Now suppose $G - e$ is not 2-connected. Then we can write (G, e) as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$ where $k \geq 2$. By Proposition 3.1.1 and by induction, we have

$$\begin{aligned}
(\delta(\overline{C}, e_C), \widehat{\delta}(\overline{C}, e_C)) &= \sum_{i=1}^{k} (\delta(\overline{B_i}, e_{B_i}), \widehat{\delta}(\overline{B_i}, e_{B_i})) \\
&\leq \sum_{i=1}^{k} (\Delta(\overline{B_i}, e_{B_i}), \widehat{\Delta}(\overline{B_i}, e_{B_i})) \\
&= (\Delta(G, e), \widehat{\Delta}(G, e)).
\end{aligned}$$

□

Checking whether (G, e) is a rooted θ-chain is equivalent to checking whether $G - \{u, v\}$ is disconnected, which can be done in linear time. More generally, we can determine the block structure of graphs with a depth first search (DFS) in $O(n(G) + |E(G)|)$ time (e.g. [**cormen2009introduction**]), which is $O(n(G))$ when G is subcubic.

Algorithm 1: `Scan`(G, e)

Input : A loop or a 2-connected subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple

Output: A half integral vector $(\Delta(G,e), \widehat{\Delta}(G,e)) \in \{(-\frac{1}{2}, \frac{1}{2}), (-1, 1), (-\frac{3}{2}, \frac{3}{2})\}$.

1 **if** *$G - e$ has a cut-vertex* **then**

2   Write (G, e) as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$;

3   **return** $\sum_{i=1}^{k}$ `Scan`$(\overline{B_i}, e_{B_i})$;

4 **if** *$G - \{u, v\}$ is disconnected or G is a loop* **then**

5   **return** $(-\frac{1}{2}, \frac{1}{2})$;

6 **else if** *$G_u - \{u_1, u_2\}$ is disconnected* **then**

7   **return** $(-\frac{3}{2}, \frac{3}{2})$;

8 **else**

9   **return** $(-1, 1)$;

Proposition 3.5.2. `Scan`(G, e) *can be computed in $O(n(G))$ time.*

Proof. If `Scan`(G, e) returns on lines 5, 7, or 9, then it performs at most three depth first searches, thus requiring $O(n(G))$ time. Now suppose `Scan`(G, e) returns on line 3; that is, (G, e) is the closure of a subcubic chain $C = xe_0B_1e_1 \cdots e_{k-1}B_ke_ky$ where $k \geq 2$. For all $i \in [k]$, $\overline{B_i} - e_{B_i}$ is either 2-connected or a single vertex, so `Scan`$(\overline{B_i}, e_{B_i})$ will not execute line 2. Thus `Scan`(G, e) requires a depth first search on an input of size $n(G)$ on line 1 and at most three depth first searches for each $\overline{B_i}$, $i \in [k]$. As $\sum_{i=1}^{k} n(\overline{B_i}) < n(G)$, we have that in all cases, `Scan`(G, e) requires $O(n(G))$ time. $\square$

We will define two algorithms `EC`(G, e) and $\widehat{\texttt{EC}}(G, e)$ which will return an even cover F in $\mathcal{E}(G, e)$ and $\widehat{\mathcal{E}}(G, e)$ respectively such that $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$ and $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$ respectively. For convenience, we wrap these two algorithms in a main algorithm `Algo` with preprocessing to handle the base case (where (G, e) is a loop) and the case where $G - e$ is not 2-connected.

Algorithm 2: $\texttt{Algo}(G, e, \text{flag})$

Input : A loop or a 2-connected subcubic graph G and $e \in E(G)$ such that $G - e$ is simple, and a binary input flag

Output: $F \in \mathcal{E}(G, e)$ such that $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$ (if flag $==$ true) or $F \in \widehat{\mathcal{E}}(G, e)$ such that $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$ (if flag $==$ false)

1 **if** G *is a loop* **then**
2   **if** *flag* $==$ *true* **then**
3   **return** $F = G$;
4   **else**
5   **return** $F = G - e$;
6 **if** $G - e$ *is not 2-connected* **then**
7   Write (G, e) as the closure $(\overline{C}, e_C)$ of a subcubic chain $C = xe_0B_1e_1B_2 \dots e_{k-1}B_ke_ky$;
8   Let $F_i = \texttt{Algo}(\overline{B_i}, e_{B_i}, \text{flag})$ for all $i \in [k]$;
9   **if** *flag* $== true$ **then**
10   **return** $F = \bigcup_{i=1}^{k}(F_i - e_{B_i}) + e + \{e_i : i \in [k-1]\}$;
11   **else**
12   **return** $F = \bigcup_{i=1}^{k} F_i$;
13 Let $(\Delta, \widehat{\Delta}) = \texttt{Scan}(G, e)$;
14 **if** *flag* $==$ *true* **then**
15   **return** $F = \texttt{EC}(G, e, \Delta)$;
16 **else**
17   **return** $F = \widehat{\texttt{EC}}(G, e)$;

For the remainder of the section, we let $f_{\texttt{Algo}} : \mathbb{N} \to \mathbb{N}$ denote a superadditive function (i.e. $f_{\texttt{Algo}}(n_1) + f_{\texttt{Algo}}(n_2) \leq f_{\texttt{Algo}}(n_1 + n_2)$ for all $n_1, n_2 \in \mathbb{N}$) such that $\texttt{Algo}(G, e, \text{flag})$ takes at most $f_{\texttt{Algo}}(n)$ steps on inputs of size at most n. We will show in the end that we can take $f_{\texttt{Algo}}(n) = O(n^2)$.

We now give the algorithm $\widehat{\texttt{EC}}(G, e)$ used in line 17 of $\texttt{Algo}(G, e, \text{flag})$, which produces an even cover $F \in \widehat{\mathcal{E}}(G, e)$ with $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$. Recall that (G_u, f_u)

is obtained from G and $e = uv$ by deleting e and suppressing u to an edge $f_u = u_1u_2$.

Algorithm 3: $\widehat{\texttt{EC}}(G, e)$

Input : A subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected

Output: An even cover $F \in \widehat{\mathcal{E}}(G, e)$ with $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \widehat{\Delta}(G, e)$ where $\widehat{\Delta}(G, e) = \texttt{Scan}(G, e)_2$

1 Let $F' = \texttt{Algo}(G_u, f_u, \text{true})$;

2 **return** $F = (F' - f_u) + \{u\} + \{u_1u, uu_2\}$;

Proposition 3.5.3. *Suppose* Algo *is correct on inputs of size less than* n*. Then* $\widehat{\texttt{EC}}$ *is correct and takes* $f_{\texttt{Algo}}(n-1) + O(1)$ *time for all inputs of size less than or equal to* n*.*

Proof. We clearly have $F \in \widehat{\mathcal{E}}(G, e)$. We claim that $\Delta(G_u, f_u) + 2 \leq \widehat{\Delta}(G, e)$. If $\widehat{\Delta}(G, e) = \frac{3}{2}$, there is nothing to prove (since $\Delta \leq -\frac{1}{2}$). If $\widehat{\Delta}(G, e) = 1$, then (G_u, f_u) is not a rooted θ-chain, so $\Delta(G_u, f_u) \leq -1$. Finally, suppose $\widehat{\Delta}(G, e) = \frac{1}{2}$. Then (G, e) is a rooted θ-chain. This implies that (G_u, f_u) is the closure $(\overline{C}, e_C)$ of a subcubic chain C with at least three blocks, so $\Delta(G_u, f_u) = \Delta(\overline{C}, e_C) \leq -\frac{3}{2}$. It follows that

$$\begin{aligned} \text{exc}(F) &= \text{exc}(F') \\ &\leq \frac{n(G) + n_2(G)}{4} + \Delta(G_u, f_u) + 2 \\ &\leq \frac{n(G) + n_2(G)}{4} + \widehat{\Delta}(G, e). \end{aligned}$$

For the time complexity, note that Algo is called only once on (G_u, f_u), which takes $f_{\texttt{Algo}}(n(G_u)) = f_{\texttt{Algo}}(n-1)$ time. The remaining lines require constant time, thus $\widehat{\texttt{EC}}$ runs in $f_{\texttt{Algo}}(n-1) + O(1)$ time. □

We now give the algorithm $\texttt{EC}(G, e, \Delta)$ in line 15 of Algo, which produces an even cover $F \in \mathcal{E}(G, e)$ such that $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \Delta(G, e) + 2$. For clarity of presentation, we split the algorithm into three cases depending on the value Δ. We first describe the case $\Delta = -\frac{1}{2}$.

Algorithm 4: $\texttt{EC}(G, e, -\frac{1}{2})$

Input : A subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected, and $\Delta(G, e) = -\frac{1}{2}$ (i.e. (G, e) is a rooted θ-chain)

Output: An even cover $F \in \mathcal{E}(G, e)$ with $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + \frac{3}{2}$

1 Determine the subcubic chains C_1 and C_2 of (G, e) with a DFS;

2 Let $(\Delta(C_1), \widehat{\Delta}(C_1)) = \texttt{Scan}(\overline{C_1}, e_{C_1})$ and let $(\Delta(C_2), \widehat{\Delta}(C_2)) = \texttt{Scan}(\overline{C_2}, e_{C_2})$;

3 Relabel if necessary so that $\Delta(C_1) + \widehat{\Delta}(C_2) \leq 0$;

4 Let $F_1 = \texttt{Algo}(\overline{C_1}, e_{C_1}, \text{true})$ and $F_2 = \texttt{Algo}(\overline{C_2}, e_{C_2}, \text{false})$;

5 Let v' be the neighbor of v in C_1 and let u' be the neighbor of u in C_1;

6 **return** $F = (F_1 - e_{C_1}) \cup F_2 + \{u, v\} + \{u'u, uv, vv'\}$;

Proposition 3.5.4. *Suppose* $\texttt{Algo}$ *is correct on inputs of size less than* $n = n(G)$*. Then* $\texttt{EC}(G, e, -\frac{1}{2})$ *is correct and takes* $f_{\texttt{Algo}}(n-1) + O(n)$ *time for all input graphs of size less than or equal to* n*.*

Proof. For correctness, first note that the relabeling step on line 3 is always possible as $\Delta(C_i) = -\widehat{\Delta}(C_i)$ for $i \in [2]$. Since $n(G) = n(\overline{C_1}) + n(\overline{C_2}) + 2$, $n_2(G) = n_2(\overline{C_1}) + n_2(\overline{C_2})$, and $\text{exc}(F) = \text{exc}(F_1) + \text{exc}(F_2)$, we have

$$\begin{aligned}
\text{exc}(F) &= \text{exc}(F_1) + \text{exc}(F_2) \\
&\leq \frac{n(C_1) + n_2(C_1)}{4} + \Delta(C_1) + 2 + \frac{n(C_2) + n_2(C_2)}{4} + \widehat{\Delta}(C_2) \\
&\leq \frac{n(G) + n_2(G)}{4} + \frac{3}{2}.
\end{aligned}$$

For the time complexity, line 1 requires $O(n)$ time. By Proposition 3.5.2, line 2 requires $O(n(\overline{C_1})) + O(n(\overline{C_2})) = O(n)$ time. By induction, line 4 takes $f_{\texttt{Algo}}(n(\overline{C_1})) + f_{\texttt{Algo}}(n(\overline{C_2})) \leq f_{\texttt{Algo}}(n-1)$ time. Thus, in total, $\texttt{EC}(G, e, -\frac{1}{2})$ takes $f_{\texttt{Algo}}(n-1) + O(n)$ time of inputs of size n. $\square$

Before we handle the analysis of $\texttt{EC}(G, e, -1)$, we first give an important subroutine which is an algorithmic version of Lemma 3.2.2.

Algorithm 5: $\texttt{Subroutine}(Z, u, v_1, v_2)$

Input : A simple 2-connected subcubic graph Z and distinct vertices u, v_1, v_2 of degree 2 in Z

Output: $F \in \mathcal{E}(Z + uv_i, uv_i)$ for some $i \in [2]$ with $\text{exc}(F) \leq \frac{n(Z+uv_i)+n_2(Z+uv_i)}{4} + 1$

```
1 For each i ∈ [2], let (Δ_i, Δ̂_i) = Scan(Z + uv_i, uv_i);
2 if Δ_i ≤ −1 for some i ∈ [2] then
3     return F = Algo(Z + uv_i, uv_i, true);
4 Let C_{i,1}, C_{i,2} denote the two subcubic chains of (Z + uv_i, uv_i), i ∈ [2];
5 Let (Δ(C_{i,j}), Δ̂(C_{i,j})) = Scan(\overline{C_{i,j}}, e_{C_{i,j}}) for i, j ∈ [2];
6 Relabel if necessary so that Δ(C_{1,1}) + Δ̂(C_{1,2}) ≤ −1/2;
7 Let F_1 = Algo(\overline{C_{1,1}}, e_{C_{1,1}}, true) and F_2 = Algo(\overline{C_{1,2}}, e_{C_{1,2}}, false);
8 Let u' be the neighbor of u in C_{1,1} and v' be the neighbor of v_1 in C_{1,1};
9 return F = (F_1 − e_{C_{1,1}}) ∪ F_2 + {u, v} + {u'u, uv_1, v_1v'};
```

Proposition 3.5.5. *Suppose* $\texttt{Algo}$ *is correct for all inputs of size less than or equal to* $n = n(Z)$*. Then* $\texttt{Subroutine}$ *is correct and takes* $f_{\texttt{Algo}}(n) + O(n)$ *time for all inputs of size less than or equal to* n*.*

Proof. We first analyze correctness. If we return on line 3, by correctness of $\texttt{Algo}$, we have $\text{exc}(F) \leq \frac{n(Z+uv_i)+n_2(Z+uv_i)}{4} + 1$. So assume $\Delta_i = \Delta(Z + uv_i, uv_i) = -\frac{1}{2}$ for both $i \in [2]$. Thus both $(Z + uv_i, uv_i)$ are rooted θ-chains, which implies that v_{3-i} is a trivial block in one of the chains $C_{i,1}$ and $C_{i,2}$. This then implies that $\Delta(C_{i,1}) \neq \Delta(C_{i,2})$ for some $i \in [2]$. Thus the relabeling step on line 6 is always possible.

Now consider the even cover F returned on line 9. As $n(Z+uv_1) = n(\overline{C_{1,1}})+n(\overline{C_{1,2}})+$

2, $n_2(Z+uv_1) = n_2(\overline{C_{1,1}}) + n_2(\overline{C_{1,2}})$, and $\Delta(C_{1,1}) + \widehat{\Delta}(C_{1,2}) \leq -\frac{1}{2}$, we have

$$\begin{aligned}
\text{exc}(F) &= \text{exc}(F_1) + \text{exc}(F_2) \\
&\leq \frac{n(\overline{C_{1,1}}) + n_2(\overline{C_{1,1}})}{4} + \Delta(\overline{C_{1,1}}) + 2 + \frac{n(\overline{C_{1,2}}) + n_2(\overline{C_{1,2}})}{4} + \widehat{\Delta}(\overline{C_{1,2}}) \\
&\leq \frac{n(Z+uv_1) + n_2(Z+uv_1)}{4} + \Delta(\overline{C_{1,1}}) + \widehat{\Delta}(\overline{C_{1,2}}) + \frac{3}{2} \\
&\leq \frac{n(Z+uv_1) + n_2(Z+uv_1)}{4} + 1.
\end{aligned}$$

For the time complexity, as $n(\overline{C_{1,1}}) + n(\overline{C_{1,2}}) < n$, lines 3 and 7 both take at most $f_{\texttt{Algo}}(n)$ time. Furthermore, by Proposition 3.5.2, the remaining lines require $O(n)$ time. Since we call exactly one of line 3 or 7, $\texttt{Subroutine}(Z, u, v_1, v_2)$ takes $f_{\texttt{Algo}}(n) + O(n)$ time. □

We are now ready to present $\texttt{EC}(G, e, -1)$.

Algorithm 6: $\texttt{EC}(G, e, -1)$

Input : A subcubic graph G and $e = uv \in E(G)$ such that $G - e$ is simple and 2-connected, and $\Delta(G, e) = -1$.

Output: $F \in \mathcal{E}(G, e)$ with $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$

Let Z_1 and Z_2 be the blocks (or single vertices) of $G - \{u, v\}$ as defined in Claim 3.3.3;

Define vertices u_i, u_i', v_j, v_j' and subcubic chains U_i, V_j for $i, j \in [2]$, as in the proof of Theorem 3.1.4;

Let $(\Delta(U_i), \widehat{\Delta}(U_i)) = \texttt{Scan}(\overline{U_i}, e_{U_i})$ and $(\Delta(V_j), \widehat{\Delta}(V_j)) = \texttt{Scan}(\overline{V_j}, e_{V_j})$ for $i, j \in [2]$;

if $Z_1 \neq Z_2$ **then**

  Relabel vertices as necessary so that $\Delta(U_1) + \Delta(V_2) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_1) \leq 0$;

  Let $Z = Z_1 \cup Z_2 \cup Y$, where Y is the subcubic chain from Z_1 to Z_2;

  Let $F_{U_1} = \texttt{Algo}(\overline{U_1}, e_{U_1}, \text{true})$, $F_{V_2} = \texttt{Algo}(\overline{V_2}, e_{V_2}, \text{true})$, $F_{U_2} = \texttt{Algo}(\overline{U_2}, e_{U_2}, \text{false})$, $F_{V_2} = \texttt{Algo}(\overline{V_1}, e_{V_1}, \text{false})$, and $F_Z = \texttt{Algo}(Z + u_1'v_2', u_1'v_2', \text{true})$;

  return $F = (F_{U_1} - e_{U_1}) \cup (F_{V_2} - e_{V_2}) \cup F_{U_2} \cup F_{V_1} \cup (F_Z - u_1'v_2') + \{u, v\} + \{u_1u, uv, vv_2\}$;

else

  Relabel vertices as necessary so that $\Delta(U_1) + \Delta(V_i) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_{3-i}) \leq 0$ for $i \in [2]$;

  Let $F_Z = \texttt{Subroutine}(Z_1, u_1', v_1', v_2')$;

  Relabel so that $u_1'v_2' \in F_Z$;

  Let $F_{U_1} = \texttt{Algo}(\overline{U_1}, e_{U_1}, \text{true})$, $F_{V_2} = \texttt{Algo}(\overline{V_2}, e_{V_2}, \text{true})$, $F_{U_2} = \texttt{Algo}(\overline{U_2}, e_{U_2}, \text{false})$, and $F_{V_1} = \texttt{Algo}(\overline{V_1}, e_{V_1}, \text{false})$;

  return $F = (F_Z - u_1'v_2') \cup (F_{U_1} - e_{U_1}) \cup (F_{V_2} - e_{V_2}) \cup F_{U_2} \cup F_{V_1} + \{u, v\} + \{u_1u, uv, vv_2\}$;

Proposition 3.5.6. *Suppose* $\texttt{Algo}$ *is correct on all inputs of size less than* $n = n(G)$*. Then* $\widehat{\texttt{EC}}(G, e, -1)$ *is correct and takes* $f_{\texttt{Algo}}(n-1) + O(n)$ *time for all inputs of size less than or equal to* n*.*

Proof. The proof of correctness follows the same structure of Section **??**. The existence of Z_1 and Z_2 follows from Claim 3.3.3, and they can be determined from the block structure of $G - \{u, v\}$ in linear time. As $\Delta(U_i) = -\widehat{\Delta}(U_i)$ and $\Delta(V_i) = -\widehat{\Delta}(V_i)$ for $i \in [2]$, the relabeling on lines 5 and 10 are always possible. Furthermore, regardless of whether $Z_1 \neq Z_2$ or $Z_1 = Z_2$, we have

- $\text{exc}(F)-2 = (\text{exc}(F_{U_1})-2)+(\text{exc}(F_{V_2})-2)+\text{exc}(F_{U_2})+\text{exc}(F_{V_1})+(\text{exc}(F_Z)-2)$,
- $n(G) = n(\overline{U_1}) + n(\overline{V_2}) + n(\overline{U_2}) + n(\overline{V_1}) + n(Z + u_1'v_2') - 2$, and
- $n_2(G) = n_2(\overline{U_1}) + n_2(\overline{V_2}) + n_2(\overline{U_2}) + n_2(\overline{V_1}) + n_2(Z + u_1'v_2') + 2$.

By induction, we have $\text{exc}(F_{U_1})-2 \leq \frac{n(\overline{U_1})+n_2(\overline{U_1})}{4}+\Delta(U_1)$, $\text{exc}(F_{V_2})-2 \leq \frac{n(\overline{V_2})+n_2(\overline{V_2})}{4}+\Delta(V_2)$, $\text{exc}(F_{U_2}) \leq \frac{n(\overline{U_2})+n_2(\overline{U_2})}{4} + \widehat{\Delta}(U_2)$, and $\text{exc}(F_{V_1}) \leq \frac{n(\overline{V_1})+n_2(\overline{V_1})}{4} + \widehat{\Delta}(V_1)$. We argue now that in both cases we have

$$\text{exc}(F_Z) - 2 \leq \frac{n(Z + u_1'v_2') + n_2(Z + u_1'v_2')}{4} - 1. \tag{3.13}$$

If $Z_1 = Z_2$, this follows from Proposition 3.5.5. If $Z_1 \neq Z_2$, then $(Z + u_1'v_2', u_1'v_2')$ is the closure of a subcubic chain with at least two blocks, namely Z_1 and Z_2. By induction on its chain-blocks, we have

$$\begin{aligned}\text{exc}(F_Z) - 2 &\leq \frac{n(Z + u_1'v_2') + n_2(Z + u_1'v_2')}{4} + \Delta(Z + u_1'v_2', u_1'v_2') \\ &\leq \frac{n(Z + u_1'v_2') + n_2(Z + u_1'v_2')}{4} - 1\end{aligned}$$

and (3.13) holds in both cases. Thus,

$$\begin{aligned}\text{exc}(F) - 2 &= (\text{exc}(F_{U_1}) - 2) + (\text{exc}(F_{V_2}) - 2) + \text{exc}(F_{U_2}) + \text{exc}(F_{V_1}) + (\text{exc}(F_Z) - 2) \\ &\leq \frac{n(G) + n_2(G)}{4} + \Delta(U_1) + \Delta(V_2) + \widehat{\Delta}(U_2) + \widehat{\Delta}(V_1) + \Delta(Z + u_1v_2', u_1'v_2') \\ &\leq \frac{n(G) + n_2(G)}{4} - 1.\end{aligned}$$

For the time complexity, note that we only call `Algo` and `Subroutine` on inputs whose sizes sum to less than n. As the remaining lines require $O(n)$ time by Proposition 3.5.2, we have that the entire algorithm requires $f_{\texttt{Algo}}(n-1) + O(n)$ time. $\square$

We now present the final case for `EC`.

Algorithm 7: $\texttt{EC}(G, e, -\frac{3}{2})$

Input : A subcubic graph G and $e = uv \in E(G)$ with $G - e$ is simple and 2-connected, and $\Delta(G, e) = -\frac{3}{2}$ (i.e. (G_u, f_u) is a rooted θ-chain)

Output: $F \in \mathcal{E}(G, e)$ with exc$(F) \leq \frac{n(G)+n_2(G)}{4} + \frac{1}{2}$

Let C_1 and C_2 denote the chains of (G_u, f_u) with common endpoints $f_u = \{u_1, u_2\}$ and $v \in V(C_1)$;

Let $x_i \in V(C_2)$ be the neighbor of u_i for $i \in [2]$;

Write $C_1 = u_1 e_0 B_1 \ldots e_{k-1} B_k e_k u_2$;

Let $\ell \in [k]$ be the unique index such that $v \in V(B_\ell)$;

Let v' denote the endpoint of $e_{\ell-1}$ in B_ℓ, and let v'' denote the endpoint of e_ℓ in B_ℓ;

Let D_1 and D_2 denote the chains of C_1 with end points $\{u_1, v'\}$ and $\{v'', u_2\}$ respectively;

For $i \in [2]$, let $(\Delta(D_i), \widehat{\Delta}(D_i)) = \texttt{Scan}(\overline{D_i}, e_{D_i})$;

Relabel if necessary so that $\Delta(D_1) + \widehat{\Delta}(D_2) \leq 0$;

Let $F_2 = \texttt{Algo}(\overline{C_2}, e_{C_2}, \text{true})$, $F_{D,1} = \texttt{Algo}(\overline{D_1}, e_{D_1}, \text{true})$, $F_{D,2} = \texttt{Algo}(\overline{D_2}, e_{D_2}, \text{false})$, and $F_\ell = \texttt{Algo}(B_\ell + v'v, v'v, \text{true})$;

return $F = (F_2 - e_{C_2}) \cup (F_{D,1} - e_{D_1}) \cup F_{D,2} \cup (F_\ell - v'v) + \{u, u_1, u_2\} + \{e_0, e_{\ell-1}, u_1 x_1, uv, uu_2, u_2 x_2\}$;

Proposition 3.5.7. *Suppose* `Algo` *is correct for all inputs of size less than* $n = n(G)$. *Then* $\texttt{EC}(G, e, -\frac{3}{2})$ *is correct and takes* $f_{\texttt{Algo}}(n-1) + O(n)$ *time for all inputs of size less than or equal to* n.

Proof. We first analyze the correctness of the returned even cover F. By induction, we have that exc$(F_2) \leq \frac{n(\overline{C_2})+n_2(\overline{C_2})}{4} + \Delta(C_2) + 2$, exc$(F_{D,1}) \leq \frac{n(\overline{D_1})+n_2(\overline{D_1})}{4} + \Delta(D_1) + 2$, exc$(F_{D,2}) \leq \frac{n(\overline{D_2})+n_2(\overline{D_2})}{4} + \widehat{\Delta}(D_2)$, and exc$(F_\ell) \leq \frac{n(B_\ell+v'v)+n_2(B_\ell+v'v)}{4} + \frac{3}{2}$. As exc$(F) - 2 = (exc(F_2) - 2) + (exc(F_{D,1}) - 2) +$ exc$(F_{D,2}) + ($exc$(F_\ell) - 2)$, $n(G) = n(\overline{C_2}) + n(\overline{D_1}) +$

$n(\overline{D_2}) + n(B_\ell + v'v) + 3$, and $n_2(G) = n_2(\overline{C_2}) + n_2(\overline{D_1}) + n_2(\overline{D_2}) + n_2(B_\ell + v'v) - 1$, we have

$$\begin{aligned}
\mathrm{exc}(F) - 2 &= (\mathrm{exc}(F_2) - 2) + (\mathrm{exc}(F_{D,1}) - 2) + \mathrm{exc}(F_{D,2}) + (\mathrm{exc}(F_\ell) - 2) \\
&\leq \frac{n(G) + n_2(G)}{4} - \frac{1}{2} + \Delta(C_2) + \Delta(D_1) + \widehat{\Delta}(D_2) + \Delta(B_\ell + v'v, v'v) \\
&\leq \frac{n(G) + n_2(G)}{4} - \frac{3}{2},
\end{aligned}$$

since $\Delta(C_2), \Delta(B_\ell + v'v, v'v) \leq -\frac{1}{2}$ and $\Delta(D_1) + \widehat{\Delta}(D_2) \leq 0$. Thus $\mathrm{exc}(F)$ satisfies our desired bound.

For the time analysis, as we only call `Algo` on inputs whose sizes sum to less than n, line 9 takes at most $f_{\texttt{Algo}}(n)$ time. Furthermore, by Proposition 3.5.2, the remaining lines require $O(n)$ time. Thus, $\texttt{EC}(G, e, -\frac{3}{2})$ takes $f_{\texttt{Algo}}(n-1) + O(n)$ time. □

To summarize, we have the following.

Corollary 3.5.8. `Algo` *is correct and takes $O(n^2)$ time.*

Proof. We show inductively that we can takes $f_{\texttt{Algo}}(n) = O(n^2)$. First note that lines 1-5 take constant time. Line 6 takes linear time to check, and if executed, lines 7-12 take $O(n) + \sum_{i=1}^{k} f_{\texttt{Algo}}(n(\overline{B_i})) \leq O(n) + \sum_{i=1}^{k} O(n(\overline{B_i})^2) = O(n^2)$.

Line 13 take linear time by Proposition 3.5.2, and in lines 14-17, we execute exactly one of $\texttt{EC}(G, e, \Delta)$ and $\widehat{\texttt{EC}}(G, e)$, which takes $f_{\texttt{Algo}}(n-1) + O(n)$ time by Propositions 3.5.3, 3.5.4, 3.5.6, and 3.5.7. It follows that we can take $f_{\texttt{Algo}}(n) = O(n^2)$. □

Corollary 3.5.9. *Given a simple 2-connected subcubic graph G, we can find an even cover F of G with* $\mathrm{exc}(F) \leq \frac{n(G)+n_2(G)}{4} + 1$ *in quadratic time.*

Proof. Pick an arbitrary edge $e \in E(G)$. Run $\texttt{Algo}(G, e, \text{true})$ and $\texttt{Algo}(G, e, \text{false})$. One of the returned even covers will have excess at most $\frac{n(G)+n_2(G)}{4} + 1$. □

Let us now complete the proof of Theorem 1.1.2, restated here for the reader's convenience.

Proof. By Corollary 3.5.9, we can find an even cover F of G with $\text{exc}(F) \leq \frac{n(G)+n_2(G)}{4}+1$ in quadratic time. Then by Proposition 1.2.2, we can convert F to a TSP walk of length $\text{exc}(F) - 2 + n(G) \leq \frac{5n(G)+n_2(G)}{4} - 1$ in linear time. $\square$

If the input graph G is cubic (i.e. $n_2(G) = 0$), then Theorem 1.1.2 finds a TSP walk of length at most $\frac{5n(G)}{4} - 1$ in quadratic time. Since every TSP walk trivially has length at least $n(G)$, this gives a $\frac{5}{4}$-approximation algorithm for TSP walks in 2-connected cubic graphs. For general subcubic graphs, Theorem 1.1.2 finds a TSP walk of length at most $\frac{3}{2}n(G)$ which trivially yields a $\frac{3}{2}$-approximation algorithm. The bound gets better for subcubic graphs with fewer vertices of degree 2; for example, if $n_2(G) \leq \frac{1}{3}n(G)$, then Theorem 1.1.2 yields a TSP walk of length at most $\frac{4}{3}n(G)$. We suspect that refining the ideas developed in this paper could lead to another $\frac{4}{3}$-approximation algorithm for subcubic graphs, matching the current best ratio by Mömke and Svensson [46], and possibly beyond.

CHAPTER 4

CONCLUSION

In Chapter 2, we developed a theory of Tutte paths where the number of possible bridges is bounded. In Chapter 3, we provided a 5/4-approximation for the cubic TSP by means of finding efficient even covers. We will now discuss possible future directions for this work.

4.1 Future Directions

There has been extensive work in extending Tutte path results to other surfaces. Thomas and Yu [59] showed that 4-connected projective-planar graphs are Hamiltonian and Kawarabayishi and Ozeki [39] later showed that such graphs are Hamiltonian-connected. Both of these results relied on Tutte path techniques. Both Grübaum [28] and Nash-Williams [49] conjectured every 4-connected graph embedded in the torus is Hamiltonian. There has been much in partial results [3, 11, 60, 61] towards this conjecture, again relying on a Tutte path strategy. It is natural to try and extend our quantitative Tutte path result to these different settings.

A *2-walk* is a spanning walk that visits each vertex at most twice. As a relaxation of the Hamiltonian cycle problem, Gao and Richter [23] showed that every 3-connected planar graph has a 2-walk. Nakamoto, Oda, and Ota [48] asked if every 3-connected n-vertex planar graph has a 2-walk such that the number of vertices visited twice is $n/3+o(1)$. Note if this were true, it would directly imply the result of Kawarabayashi and Ozeki [40] on the length of tsp walks in 3-connected planar graphs up to to additive constant error. Later, Gao, Richter, and Yu [24] showed that every 3-connected planar graph has a 2-walk, such that any vertex visiting twice is contained in a 3-cut. This was developed by developing a system of distinct representatives (SDR) between a Tutte path and its bridges. It would be interesting if this SDR can be developed simultaneously while bounding the number of nontrivial bridges to be $n/3 + o(1)$. This could have potential applications in answering Nakamoto, Oda, and Ota's question.

From an algorithm's perspective, our approximation algorithm for cubic TSP is unsophisticated as it simply provides a walk of length $5n/4-1$ where n is the number of vertices. As every TSP walk has length at least n, this provides the $5/4$-approximation guarantee. To push the $5/4$-approximation guarantee further, a possible strategy is to develop efficient means to compute better lower bound guarantees on the optimal walk. This seems challenging, as this lower bound guarantee cannot be used to detect if a graph is Hamiltonian or not, as the Hamiltonian cycle problem remains NP-hard even when restricted to 3-connected cubic planar graphs [25].

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